AD NUMBER
AD480899
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 1962. Other requests shall be referred to U.S. Naval Postgraduate School, Monterey, CA 93943.**
AUTHORITY
USNPS ltr, 27 Sep 1971

SERVOMECHANISM ANALYSIS OF A TWO-RATE SPRING AUTOMOBILE SUSPENSION

P. G. FOSSUM

SERVOMECHANISM ANALYSIS

OF A TWO-RATE SPRING AUTOMOBILE SUSPENSION

* * * * *

P. G. Fossum

SERVOMECHANISM ANALYSIS

OF A TWO-RATE SPRING AUTOMOBILE SUSPENSION

by

P. G. Fossum

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
Monterey, California

1 9 6 2

SERVOMECHANISM ANALYSIS

OF A TWO-RATE SPRING AUTOMOBILE SUSPENSION

BY

P. G. Fossum

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

from the

United States Naval Postgraduate School

ABSTRACT

Some of the methods of automatic control are used to investigate the feasibility of a suspension system using a spring with two spring rates. Following a disturbance the spring rate is changed discontinuously from one value to the other in such a manner that a damping action results. A single degree of freedom corresponding to the vertical motion of the automobile body with respect to one wheel is considered and this system is simulated on the analog computer. The objectives of this investigation are: (1) to determine the effect of the spring in reducing the frequency response resonance peak. (2) to determine the transient response characteristics of the system when subjected to a step disturbance. The results show that the two-rate spring does provide a significant amount of damping.

The writer wishes to express his appreciation for the suggestion of this thesis topic and for the encouragement and assistance given him by Doctor G. J. Thaler of the Electrical Engineering Department of the U. S. Naval Postgraduate School.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

List of Illustrations (Continued)

TABLE OF SYMBOLS

SYMBOL	DESCRIPTION
A	Input signal amplitude (sine wave or step)-feet
B	Spring-rate return - per cent or pounds per foot
C	Viscous damping constant - pounds per foot per second
E,e	Position error (A measure of spring compression or tension) feet
$\dot{E}$	Rate of change of position error - feet per second
f	Frequency - cycles per second
F_c	Viscous damping force - pounds
F_k	Spring force pounds
K	Spring rate - pounds per foot
M	One-fourth of the spring mass - slugs
S	LaPlace Operator
t	Time - seconds
X_1	Input displacement (wheel displacement) - feet
X_2	Output displacement (sprung mass displacement) - feet
$\dot{X}_2$	Output velocity - feet per second
ζ	Damping ratio of linear system
ω	Radian frequency - radians per second
ω_n	Natural frequency
ω_r	Resonant frequency

CHAPTER I

INTRODUCTION

Following the acceptance of the automobile as a reliable means of transportation manufacturers have continually striven to improve the suspension system to increase passenger safety and comfort. The obvious method of determining the ride characteristics of a proposed suspension is to install the suspension and drive and ride in the automobile over various types of roads to observe its performance. To determine what changes should be made one makes the changes and observes their influence on the ride characteristics. This is the observation method of ride analysis and this method was largely used in evolving the suspension system of todays automobiles.

However with the observation method the observer can only observe the characteristics of one automobile at a time and therefore he must rely on his memory to compare one automobile suspension with the next. In other words the observation method is subjective and it does not furnish numerical data which can be evaluated and compared over long-time periods.

Mathematical analysis using an analog computer can provide numerical data and the means for comparing and evaluating the characteristics of proposed designs. If this analysis proves a particular design to be feasible, then of course, the subjective observation method can be used. Often however, the time and expense associated with fabricating the suspension and conducting the observation test can be saved when the mathematical analysis proves the design not feasible. There is another advantage to the mathematical analysis. The differential equations that

describe the auto suspension system are of the same form as those of a position servomechanism. This suggests that the techniques of automatic control might be used in analyzing and improving the ride characteristics of automobile suspension systems.

This thesis presents the results of a mathematical study of a suspension system in which the usual constant rate spring is replaced with a non-linear spring; i.e., a spring with a spring rate that is switched to either of two values. Switching is a function of spring displacement and rate of change of spring displacement. Investigation is restricted to one degree of freedom corresponding to the vertical displacement of the auto body with respect to the wheels. From the system differential equations a block diagram is formed similar to a servomechanism and fundamental automatic control techniques are used to obtain the closed loop frequency response on the analog computer. Data is obtained for systems employing the non-linear spring with and without a viscous damping shock absorber. The effect of varying the viscous damping is investigated and also the effect of changing the ratio of the two values of spring rate.

Chapter II presents the justifications for treating the suspension as a position servomechanism, and develops the control type block diagram that represents the automobile suspension. The defining relationships for the two-rate spring are given together with the reasons for considering such a suspension spring.

In order to compare the proposed suspension with a conventional suspension system the characteristics of a typical automobile suspension are explained. A summary of the literature on human comfort criteria is included and compared with the typical suspension system to determine how the suspension characteristics should be changed to improve the riding quality.

For a linear approximation to the behavior of the two-rate spring the describing function is derived from a Fourier analysis of the spring force output wave form resulting from a sinusoidal displacement function.

To simulate the proposed suspension system on the analog computer is was necessary to devise a means of simulating the two-rate spring. Whereas analog computer literature contains methods for simulating many common non-linearities none of these techniques are suitable for simulating the two-rate spring. Chapter V describes the special transistor/relay logic circuit designed to perform the spring rate switching operation.

Chapter VI and VII, present the results of frequency response and transient response tests respectively. Transient response data is displayed in the phase portrait form.

The final chapter summarizes the findings of the feasibility study and suggests further suspension system research.

CHAPTER II

THE AUTOMOBILE SUSPENSION AS A SERVOMECHANISM

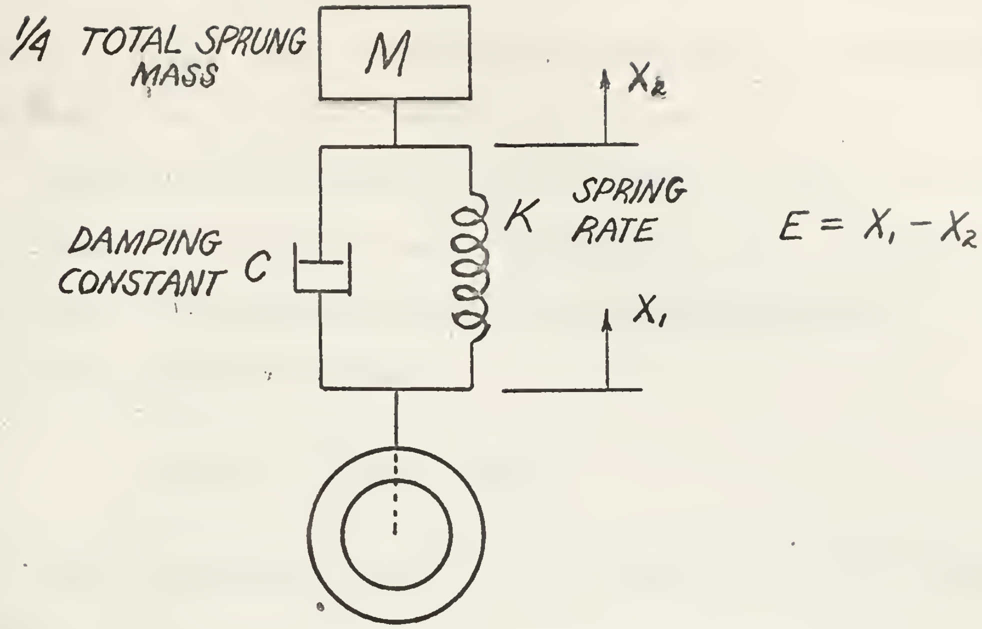
The first applications of mathematical analysis to automobile suspension system performance simply consisted of solving the differential equations of motion and determining how stability and ride characteristics were related to the coefficients and form of the equations. The characteristics of individual system components and the constants representing physical parameters do not appear explicitly in the equations and therefore the solutions to the differential equations of motion do not readily indicate how the physical system should be changed to improve the dynamic characteristics of the suspension. Whereas this method of mathematical analysis can be used to evaluate the characteristics of a particular suspension, trial and error techniques must be used to design system improvements.

The pioneer automatic control systems engineer was faced with this same design handicap. The differential equations describing a control system do not directly reveal what changes must be made to the physical system to obtain a desired change in the response. For simple control systems trial and error methods can be used to ultimately obtain the desired response. However, as control specifications demand greater speed and/or greater accuracy the system must grow in size and complexity to include more system components, and as the system becomes more complex it becomes increasingly more difficult to apply trial and error techniques to solve the design problem. During World War II military requirements for high performance such as in anti-aircraft fire control systems resulted

in control systems that were so complex that trial and error techniques were no longer practical.

To facilitate the design of these complex systems there was developed an operational method of mathematical analysis employing La Place transforms. It is still a method of solving the differential equations of motion but an advantage results from a different approach to the solution. Emphasis is placed on the physical arrangement of the system and the effect of individual components on overall system performance is more readily determined. The operational approach to the analysis makes possible a block diagram representation of the system that is also useful in relating overall system performance to the characteristics (La Place transform functions) of individual components. Numerous other control design devices are made available through the application of La Place transforms to the differential equations of the system. A detailed explanation of this operational transfer function approach and associated techniques is available in any feedback control systems text and details will not be given here.

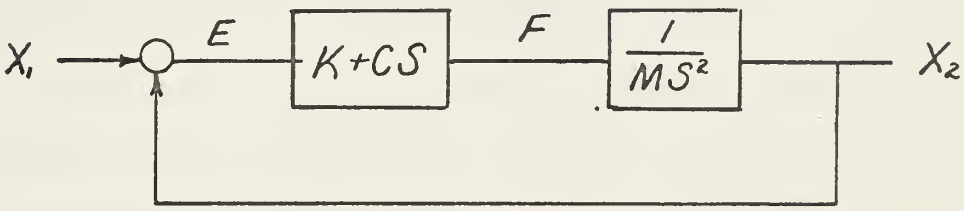
The similarity of the differential equations of motion for the automobile suspension system to those for a position servomechanism suggests that the operational method and associated feedback control techniques might be used for analysis of suspension systems. A conventional auto suspension system and the equivalent block diagram developed from the transformed differential equations are shown in Figure 2-1. Of course, the complete suspension system is a five mass system (body plus four wheels) and if all modes of translation and rotation are considered for each mass the system then has 30 degrees of freedom. However, ride characteristics and stability are predominantly dependent upon vertical



SPRING FORCE $\quad F_K(S) = K[X_1(S) - X_2(S)] = KE(S)$

DAMPING FORCE $\quad F_C(S) = CS[X_1(S) - X_2(S)] = CSE(S)$

ACCELERATING FORCE $\quad F(S) = F_K(S) + F_C(S) = MS^2X_2(S)$



$$\frac{X_2}{X_1} = \frac{K+CS}{MS^2+CS+K} = \frac{K/M\left(1 + C/K\,S\right)}{K/M\left(1 + C/K\,S + M/K\,S^2\right)}$$

Fig. 2-1 The Automobile Suspension as a Servomechanism

displacement of the auto body and therefore this thesis work is confined to a single degree of freedom corresponding to the vertical displacement of the body with respect to one wheel.

The differential equations for the system of Figure 2-1 are stated as follows. For a vertical wheel disturbance, $X_1(t)$, a spring force, $F_k(t)$, will result proportional to the relative displacement of the body with respect to the wheel.

$$F_K(t) = K[X_1(t) - X_2(t)] \tag{2-1}$$

The viscous damping action of the shock absorber develops a damping force $F_c(t)$:

$$F_c(t) = C[\dot{X}_1(t) - \dot{X}_2(t)] \tag{2-2}$$

The combined forces accelerate the body, M:

$$F_K(t) + F_C(t) = M\ddot{X}_2(t) \tag{2-3}$$

Substituting (2-1) and (2-2) in (2-3):

$$K[X_1(t) - X_2(t)] + C[\dot{X}_1(t) - \dot{X}_2(t)] = M\ddot{X}_2(t) \tag{2-4}$$

To transform Equation (2-4) to La Place operational notation initial condition values of $X_2(t)$ and $\dot{X}_2(t)$ must be known. If the system is initially in vertical static equilibrium then:

$$X_1(0) = X_2(0) = \dot{X}_2(0) = 0$$

After transformation Equation (2-4) becomes:

$$K[X_1(s) - X_2(s)] + C[SX_1(s) - SX_2(s)] = MS^2X_2(s)$$

$$(K + CS)[X_1(s) - X_2(s)] = MS^2X_2(s) \qquad (2\text{-}5)$$

Position error is defined as the relative displacement of the wheel with respect to the body.

$$E(s) = X_1(s) - X_2(s) \qquad (2\text{-}6)$$

From the transformed equations the equivalent block diagram is easily developed and the similarity to a position servomechanism becomes apparent. The input and output of this servo-like system are the vertical displacements of the wheel and body respectively. E corresponds exactly to the position error of a servomechanism. This error is acted on by the system gain, K/M, and the transfer function 1 + C/K S, until the error is reduced to zero. Note that if the shock absorber is removed C equals zero and the system then lacks a device for dissipating energy. Following a vertical disturbance imparted to the wheel, the energy introduced remains within the system and the body, M, will continue to oscillate as the energy is transferred back and forth between the potential energy of the spring and the kinetic energy of the moving body mass, M. To prevent this sustained oscillation the viscous damper shock absorber is included in present automobile suspensions.

In the suspension system unlike the usual servomechanism, it is not required or desired that the output closely follow the input. Large "position errors" are generally desired and slow follow up is permitted

so that the passenger is not subjected to displacements and accelerations corresponding to each and every imperfection when driving over washboard and pot-hole road surfaces. In other words it is desired that the output of the suspension system follow some general average trend of the input. From servo theory it is known that this low fidelity can be obtained by making system gain, K/M, small which means a weak spring and/or a large body mass. However, as the spring constant is decreased the tendency for the spring to "bottom" becomes greater, i.e., for a large disturbance the amplitude of oscillations exceeds the range of spring compression or the clearance between chassis and body. When this occurs any further wheel displacement is transmitted directly to the body with the obvious detrimental effect on passenger comfort, and perhaps structural damage will occur.

If K/M must be small for low fidelity considerations then the tendency to bottom must be controlled by viscous damping, or conceivably by some other scheme not yet developed. The trend in automobile suspensions over the last two decades has in fact been toward ever decreasing K/M necessitating ever increasing shock absorber viscous damping. However, viscous damping introduces adverse characteristics with the useful amplitude damping. In the transfer function K+CS it is noted that the shock absorber is a velocity sensitive element. Damping force, $F_c(s)$, is proportional to CSE where S is the La Place notation for the first derivative. Therefore, even small amplitude disturbances when suddenly applied cause large accelerating forces.

To summarize, a suspension employing linear viscous dampers and springs cannot provide ride characteristics that are optimum in all respects. Compromises must be made and usually the least satisfactory character of the ride is the disturbance transmitted to the passenger through the

velocity sensitive shock absorber when the automobile passes over step-like pot-hole road imperfections.

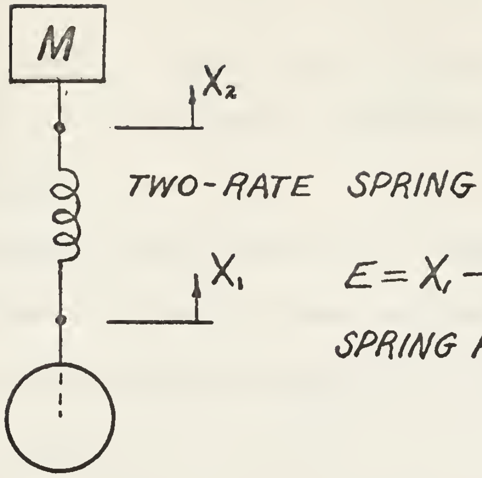
The insight gained by a control systems approach to suspension analysis suggested to Martin and Jeska[1] that one means of diminishing passenger disturbance when riding over pot-holes etc., is to use a non-linear viscous damping action. The system they investigated incorporates a large and a small viscous damping rate. The low viscous damping rate is effective when the suspension is in vertical static equilibrium and remains effective when a step disturbance is applied. Initial accelerating forces transmitted to the passenger are thus minimized. Following the step disturbance the large viscous damping rate is introduced to quickly dampen oscillations.

Another suggestion was made by Professor A. G. Thompson of the University of Adelaide, Adelaide, South Australia. Based on the control systems approach Thompson proposes a suspension system in which the viscous damper is omitted and oscillation damping is provided by a two-rate spring. When a disturbance sets the system into oscillation the spring rate is switched between a small value and a large value. The switching points are determined by position error and rate of change of position error.

Figure (2-2) illustrates the switching action. Briefly stated whenever position error and rate of change of position error have like signs then the large spring rate is switched in and when position error and rate of change of position error have opposite signs the small spring rate is switched in. For a sinusoidal position error there will be four switching points each period. The large spring rate switches in as position error

---

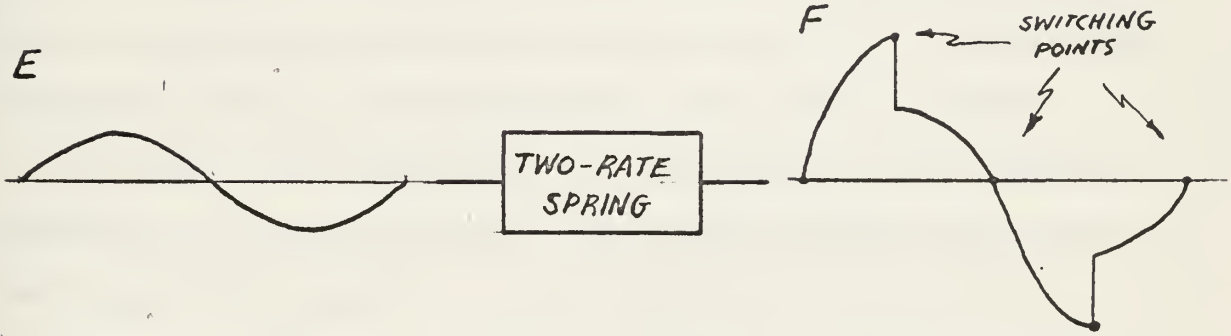
[1] G. J. Martin, and R. D. Jeska, IRE Convention Record, Pt I, pp 89-98, 1953.



$$E = X_1 - X_2$$

$$\text{SPRING RATE} = K \quad \text{WHEN } \frac{E}{\dot{E}} > 0$$

$$= K - B \quad \text{WHEN } \frac{E}{\dot{E}} < 0$$



1 ST QUARTER-CYCLE

E AND $\dot{E}$ POSITIVE ; $\frac{E}{\dot{E}} > 0$; $F = KE$

2 ND QUARTER-CYCLE

E POSITIVE, $\dot{E}$ NEGATIVE ; $\frac{E}{\dot{E}} < 0$; $F = (K-B)E$

3 RD QUARTER-CYCLE

E AND $\dot{E}$ NEGATIVE ; $\frac{E}{\dot{E}} > 0$; $F = KE$

4 TH QUARTER-CYCLE

E NEGATIVE, $\dot{E}$ POSITIVE ; $\frac{E}{\dot{E}} < 0$; $F = (K-B)E$

Fig. 2-2 Defining Relations for the Two-Rate Spring

passes through zero and the small spring rate switches in as rate of change of position error passes through zero. Because switching occurs at quarter cycle intervals, Thompson calls this switching action, quarter cycle return.

Thompson's idea is based on a fundamental method of compensating a servomechanism to make it less oscillatory. If a network or a device is inserted in the forward signal path to introduce a phase advance it can be seen by examining the transfer function of the device that the phase advance accomplishes an approximate differentiation. The phase lead operation is effective therefore in damping out oscillations but hopefully in the automobile suspension the approximate differentiation will not transmit high accelerating forces to the passenger when step inputs are applied.

For quarter cycle spring return it is obvious that the fundamental component of the spring force wave form is phase advanced with respect to the position error signal.

Figure (2-3) presents a more intuitive explanation of the damping action provided by spring rate switching. The upper panel of the figure shows a sketch of position error versus time for a step input disturbance. The lower panel schematically depicts the condition of the spring (stretched or compressed) and which spring rate is effective at times corresponding to those labeled on the error sketch. A study of the schematic in conjunction with the error sketch reveals that whenever the body mass, M, is moving away from steady state position the large spring rate is effective and when the body moves toward steady state the small spring is effective. At extremes of error all system energy is contained in the spring which is either in maximum compression or maximum tension. As maximum compression or tension is approached the large spring rate is

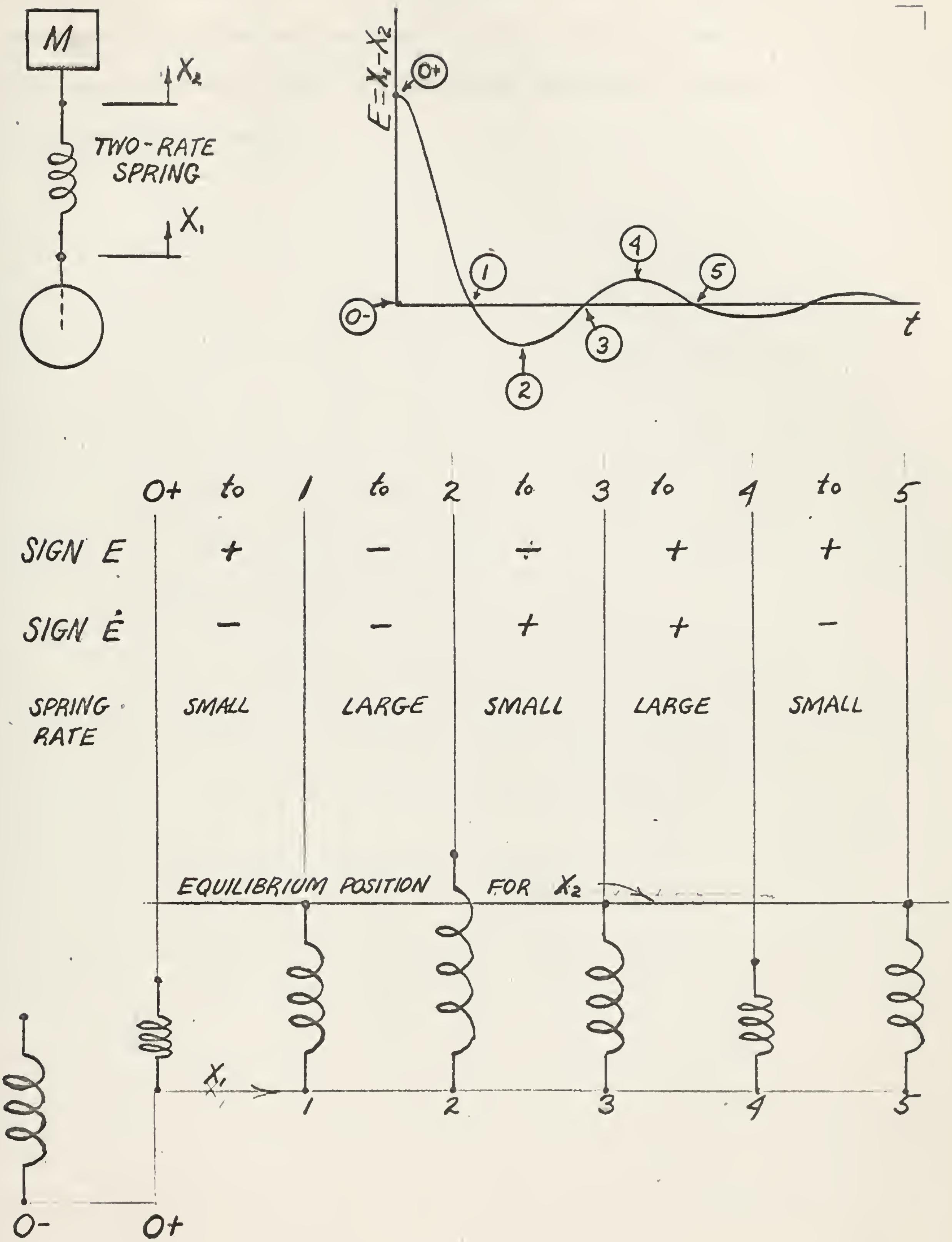


Fig 2-3 Switching Sequence Following a Step Disturbance

effective. Switching occurs at the maximum condition and part of the energy is dissipated when the small spring rate is switched in. On each successive swing there is less energy available to transfer to kinetic energy of motion.

CHAPTER III

PASSENGER COMFORT CRITERIA

The previous chapter mentions one major difference between the suspension system and the usual servo-mechanism. Servomechanism specifications normally require that the output position closely follow the input position, whereas in the suspension system large position errors are desired to give the passenger a smooth ride over road imperfections. Stated another way it is desired that the verticle motion of the auto body only follow some average trend of the road surface profile. The difficulty in determining specifications for an optimum suspension system partly lies in the interpretation of what is to be considered the average trend for passenger comfort purposes.

Measurement of the subjective effects of vertical motion is complicated by psychological effects and physical effects caused by other factors such as air temperature and humidity, engine noise, and other less tangible effects that contribute to the passenger's sense of security and well-being. Comfort in its broadest sense is a measure of the passenger's freedom from both mental and physical disturbance and it is not easy to completely separate discomfort caused by vertical motion from discomfort from other sources. Also, threshold of discomfort varies between men and women, with age, and even between two persons of the same age and sex.

Although the experimental work of this thesis is concerned only with vertical motion in a system with a single degree of freedom, a complete study must consider passenger disturbances acting in a horizontal plane and the various rotational motions associated with cornering and braking.

Considering only vertical disturbances the ride motion can be separated into three frequency bands as shown in Figure (3-1).[2,3,4] The lower frequency band extends from zero to about one cycle per second for a typical automobile. In this range there is no ride smoothing. The wheel and the body both follow the road surface profile. Ride quality may be improved by narrowing this frequency band but there must always be a low band within which no smoothing is provided so that the vehicle can follow the road contours of humpback bridges, large dips, and sine waves with crest to crest distances large with respect to the wheel base.

The mid-frequency band extends up to about two cycles per second and includes the natural frequency of the body mass/spring system. In the closed loop transfer function of Figure (2-1), the constant term in the denominator determines the natural body frequency.

$$\omega_n^2 = K/M \tag{3-1}$$

For disturbances in the mid range the wheel follows the road contour and the body mass/spring combination amplifies the vertical disturbance and so the body oscillates with greater amplitude than that of the road imperfections. The viscous damper shock absorber reduces this resonance peak but the amount of damping required to completely eliminate the peak will make the step response characteristic unacceptable.

---

[2] H. L. Cox, The Riding Qualities of Wheeled Vehicles, Proceedings of Auto Division of Inst. of Mech Engineers, 1955-56.

[3] David Hodkin, Progress in Comfort, Part One, The Autocar, pp 747-750, May 12, 1961.

[4] R. Schilling and H. O. Fuchs, Modern Passenger-Car Ride Characteristics, J. Appl. Mech., 8, pp A-59-65, June, 1941.

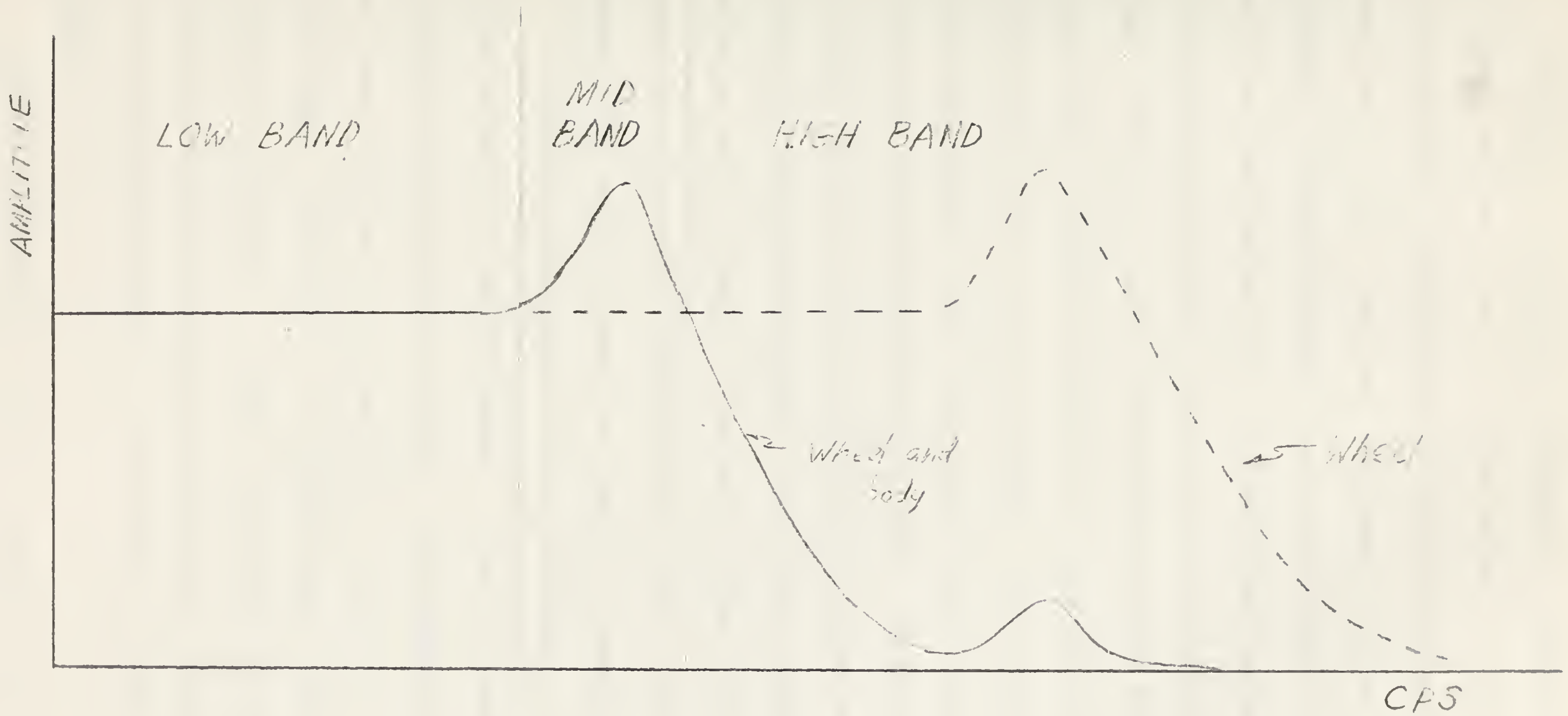


Fig. 3-1 A Typical Suspension System Frequency Response

The body is essentially unaffected by road disturbances in the upper frequency band above two cycles per second. In this band the filtering action of the body mass/spring combination attenuates the amplitude of wheel oscillations. The wheel continues to follow the road contour when the disturbing frequency falls in the lower end of this band. However, for disturbances with a frequency near the natural frequency of the unsprung mass between the tire and the suspension the amplitude of wheel oscillation exceeds the amplitude of road surface contour. This natural wheel resonance frequency is a function of the unsprung mass and the spring rate of the tire and equals about eight cycles per second for a typical automobile. The viscous damper shock absorber reduces this wheel resonance but even without damping, wheel resonance is normally sufficiently beyond the pass band of the suspension system so that passengers do not feel wheel resonance vibrations. (Also wheel resonance normally occurs below the audio threshold.) For still higher frequencies ride smoothing action is accomplished entirely by the tires. The tire follows the road imperfection while the wheel and body ride smoothly.

In the above discussion we have considered the automobile suspension to be excited by a single frequency. That is, constant speed over uniformly spaced ripples has been assumed. The frequency then is proportional to the speed and inversely proportional to the crest to crest separation or wave length. Occasionally a uniformly spaced ripple or washboard road surface is encountered. However road surface imperfections are usually randomly spaced with varying amplitudes, and consequently the suspension is simultaneously subjected to disturbances falling in all three frequency bands.

With the fundamental suspension characteristics established qualitatively in terms of a frequency spectrum, physiological response to

sinusoidal motion will now be considered. A search of the literature revealed numerous articles on the subject of human comfort criteria for harmonic motion. Although at first glance one might expect a comfort criteria for sudden displacements such as step functions, the complete "passenger suspension" system includes the seat cushions in addition to the auto suspension. Therefore to subject the passenger to a vertical step input, the automobile should be dropped from a tall building or be subjected to some other unlikely disturbance. Two of the more complete quantitative treatments of vertical vibration human comfort criteria will be summarized.

Lippert, a research analyst for Douglas Aircraft Company, has reconciled graphically the findings of a number of studies on the effect of vertical vibration on human comfort.[5] Figure (3-2) presents his daca for thresholds of perception and discomfort. It is likely a common experience that the comfort reaction is predominately related to acceleration and jerk, i.e., to the second and third derivative of displacement. For simple harmonic motion the displacement is of the form:

$$X = A \sin 2\pi f t \qquad (3-2)$$

The first, second, and third derivatives are respectively

$$\text{velocity} = \dot{X} = 2A\pi f \cos 2\pi f t \qquad (3-3)$$

$$\text{acceleration} = \ddot{X} = -4A\pi^2 f^2 \sin 2\pi f t \qquad (3-4)$$

$$\text{jerk} = \dddot{X} = -8A\pi^3 f^3 \cos 2\pi f t \qquad (3-5)$$

Because acceleration and jerk are proportional to the second and third power of frequency respectively by intuition we would expect threshold amplitude to decrease with increasing frequency as found on Figure (3-2).

---

[5] Lippert, Human Response to Vertical Vibration, SAE Journal, 55, pp 32-34, May 1947.

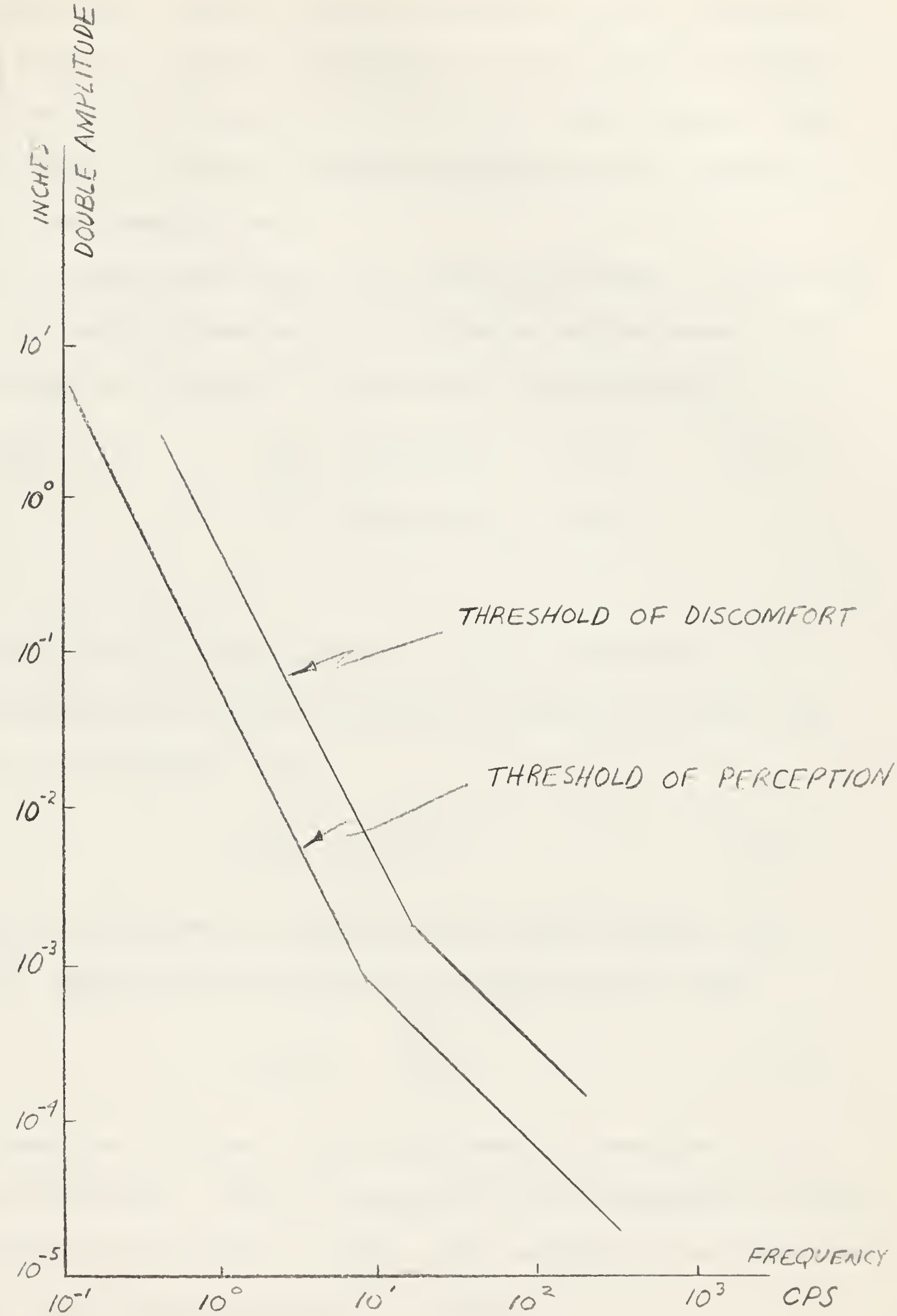


Fig. 3-2 Threshold of Discomfort vs Vertical Vibration Frequency

Janeway, Head, Dynamics Research Department, Chrysler Corporation has made a similar study and presents his data in a slightly different manner.[6] The vibration spectrum is divided into three frequency bands in each of which a different measureable characteristic of vibration determines the threshold level.

The low range extends from 1 to 6 cycles per second. In this range jerk is the limiting characteristic of vibration and the comfort limit is established at 40 Ft/Sec.^{3} An equivalent limit expression is:

$$Af^3 \leq 2 \quad ; \quad \begin{array}{l}(A = \text{amplitude in inches}) \\ (f = \text{frequency in cps})\end{array} \qquad (3\text{-}6)$$

In the mid-range extending from 6 to 20 cps acceleration is the limiting characteristic and the threshold of discomfort is established at 3.3% g = 1.07 Ft/Sec.^{2} or:

$$Af^2 \leq 1/3 \qquad (3\text{-}7)$$

Velocity determines the threshold in the high frequency range, 20 to 60 cps. Maximum velocity amplitude is .105 inches per second.

$$Af \leq 1/60 \qquad (3\text{-}8)$$

A comparison of Lippert's data with Janeway's shows that they are reasonably consistent. Lippert's piecewise linear representation places the threshold of discomfort at slightly lower amplitude values than does Janeway's closed form relationship, in Equations (3-6,7,8).

---

[6] R. N. Janeway, Passenger Vibration Limits, SAE Journal, 56, pp 48-49, August 1948.

Comfort criteria can now be correlated with suspension system frequency characteristics of Figure (3-1). Body resonance falls within the jerk sensitive range of comfort. To improve the frequency response in this range, it is desired to reduce the resonance peak and lower the resonant frequency.

Wheel resonance falls within the acceleration sensitive range. Here also it appears desirable to lower the resonant frequency to take advantage of a higher discomfort threshold. However wheel resonance must be kept beyond the suspension band-pass cutoff.

The comfort criteria presented in this chapter are used in the experimental work as a means of comparing the two-rate spring suspension with the normal suspension.

CHAPTER IV

DESCRIBING FUNCTION FOR THE TWO-RATE SPRING

The suspension system investigated has been made intentionally non-linear by including a spring with a spring rate discontinuously switched between a larger and a smaller value. This makes the differential equations of the system non-linear. Likewise the transfer function of the spring is non-linear. Usually non-linearities make the differential equations unsolvable by classical mathematical methods and automatic control methods used on linear systems are often awkward or not applicable. This chapter treats the non-linear spring by means of describing function theory,[7] which is a control systems method developed to extend linear transfer function techniques and frequency response techniques to systems containing a non-linear component.

Briefly stated a describing function is an approximate linear transfer function used to represent a non-linear component. If the approximation is a good one then all of the linear system frequency response techniques can be used to analyze the non-linear system. The describing function relates the fundamental component of output waveform of the non-linearity to the input. The following assumptions are made. (1) the input waveform is a pure sine wave; (2) the output waveform can be approximated satisfactorily by the fundamental component of the Fourier series representation; (3) no zero frequency component or sub-harmonics are present.

---

[7] G. J. Thaler and M. P. Pastel. Analysis and Design of Non-linear Control Systems. McGraw-Hill Book Company, Inc., New York, 1962.

For the two-rate spring non-linearity the spring force output waveform is conveniently expressed by a Fourier series. Therefore, the describing function is obtained simply by taking the quotient of the fundamental term of the Fourier series and the input sine wave. To write the Fourier series for the output waveform it is helpful to express the defining relationships of Figure (2-2) in another form:

$$K \equiv \text{larger spring rate}$$

$$K-B \equiv \text{smaller spring rate}; \quad 0 < B < K$$

Then the defining equations are:

$$F(s) = K\,E(s) \quad \text{when} \quad E/\dot{E} > 0 \tag{4-1}$$

$$F(s) = (K-B)\,E(s) \quad \text{when} \quad E/\dot{E} < 0 \tag{4-2}$$

Here B is a measure of the amount of spring rate return, i.e., the amount by which the larger rate is reduced by switching. Using Equations (4-1) and (4-2) to obtain the output Fourier series the describing function, G_d, is found to be:*

$$G_d = \sqrt{\left(B/\pi\right)^2 + \left(K - B/2\right)^2}\;\epsilon^{j\psi} \tag{4-3}$$

$$\psi = \arctan \frac{B}{\pi(K-B/2)} \tag{4-4}$$

*The describing function is derived in Appendix I

In general, describing functions may be used to predict the closed loop frequency response and to investigate system stability. In this thesis work the describing function is used only as a preview to the behavior of the two-rate spring. Equation (4-3) confirms the suspicion that the two-rate spring transfer function is insensitive to input amplitude and frequency. For the frequency response experimental procedure this means that the input signal to the spring transfer function, G_d, need not be held constant as the input frequency is varied to produce the frequency response data. Equation (4-4) indicates that the two-rate spring introduces a constant phase advance independent of frequency. Recall that it is Thompson's desire to introduce such a phase shift to create an approximate derivative effect. Equation (4-4) also shows that the linear approximate transfer function produces the maximum phase advance for 100% spring return, i.e., for the limiting case B equals K for a maximum phase advance of arctan $2/\pi = 32.6$ degrees.

CHAPTER V

ANALOG SIMULATION OF THE TWO-RATE SPRING

To simulate the proposed two-rate spring suspension on the analog computer it was first necessary to find a technique for simulating the switch action of the two-rate spring. In the language of the computer the gain of an operational amplifier must be increased and decreased discontinuously as prescribed by the defining equations of the spring. Whereas analog computer texts and the literature describe techniques for simulating the common non-linearities such as saturation, relay dead-zone, relay hysteresis etc., none of these techniques are suitable for the two-rate spring. This chapter describes the circuit designed to perform the switching operation.

A written statement of the switching required suggests the use of logic circuits.

> When position error <u>and</u> rate of change of position error are both positive <u>or</u> when position error <u>and</u> rate of change of position error are both negative then the larger spring-rate is effective.

Two "<u>and</u>" logic circuits and one "<u>or</u>" logic circuit are required to determine switching conditions. Amplifier gain can be easily changed by switching in a resistor. Therefore, a parallel combination of two relay contacts can be used for the <u>or</u> logic. If normally open relay contacts are used any of the four arrangements in Figure (5-1) can be used. The particular "<u>and</u>" logic circuits chosen require that either (4-1-b) or (4-1-c) be used to provide the larger amplifier gain when the "<u>or</u>" conditions are fulfilled.

Diode "<u>and</u>" logic circuits were first tried for operating the "<u>or</u>"

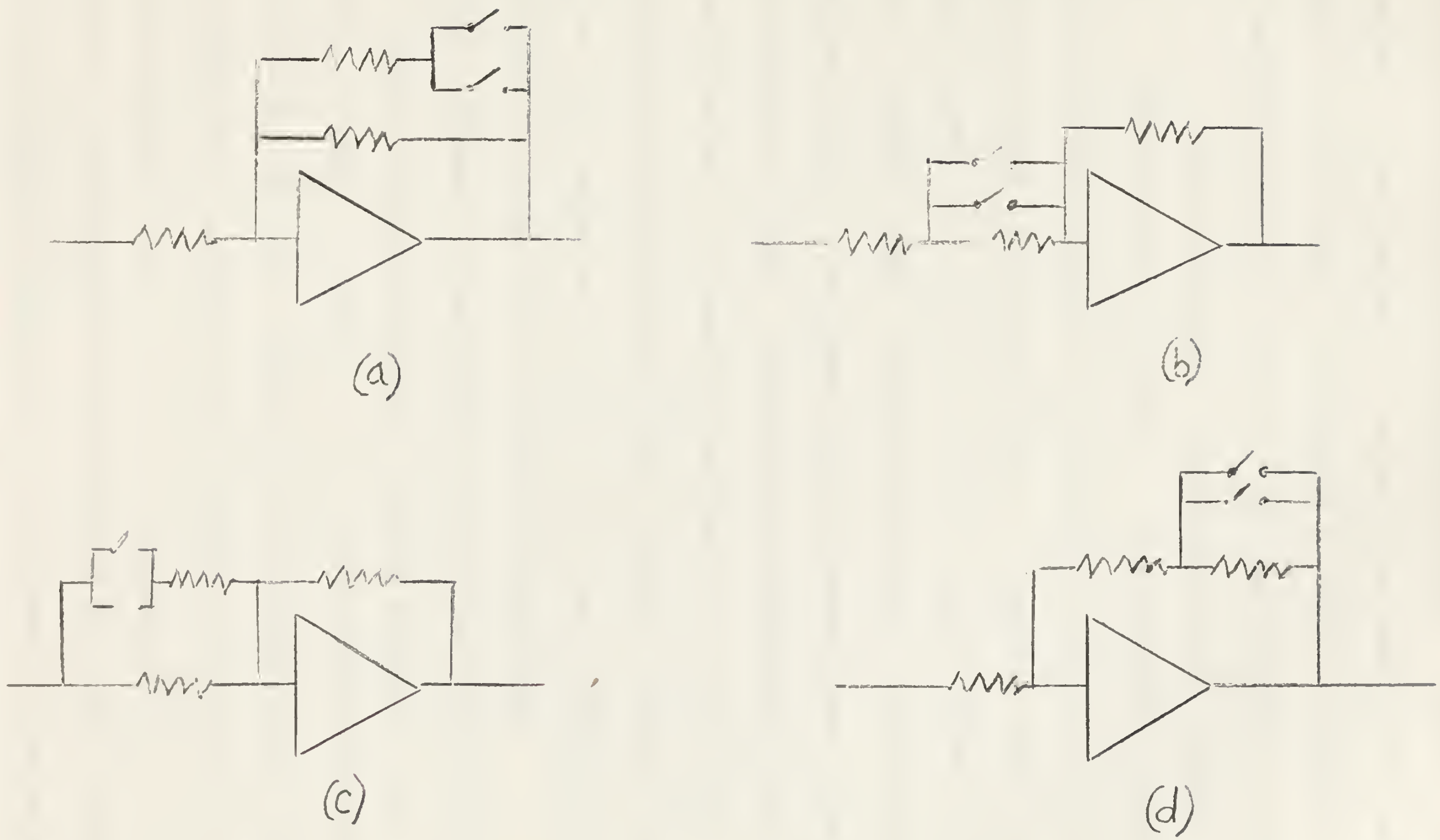


Fig. 5-1 Relay "OR" Logic

relays. However the diode "and" circuit output current level is the same as the input signal current level which in this case is only the current available from an operational amplifier. Pull-in current for the available relays exceeds the output of an operational amplifier and therefore the diode logic was abandoned in favor of transistor logic.

A schematic of the transistor "and" logic circuits is shown in Figure (5-2). Each relay coil is operated by a two transistor "and" circuit. PNP transistors are used in a common emitter configuration. Both collectors are tied together and connected to one end of the relay coil. Each "and" circuit operates as follows. A positive input to the base of a transistor turns that transistor off and a high resistance appears in the emitter - collector path. A negative base input turns that transistor on and emitter-collector resistance drops to several ohms effectively clamping both collectors to ground. Therefore both transistors must be turned off with positive inputs to energize the relay. As a result when E and $\dot{E}$ are both positive, relay number one closes, and when $-E$ and $-\dot{E}$ are both positive then relay number two closes. In this way spring-rate switching is accomplished as called for by the defining equations.

Some refinement is necessary to allow correct relay switching independent of position error amplitude. (Position error for a step input is a decaying sine wave oscillation). It is desired that each transistor turn fully on, i.e., saturate, whenever the base signal passes through zero going negative and conversely each transistor should turn off immediately when the base signal passes through zero going positive. Therefore it is necessary to use an amplifier with high gain and diode limiting to produce an approximately square position error wave. Likewise rate of change of position error is produced with a diode-limited high gain differentiating

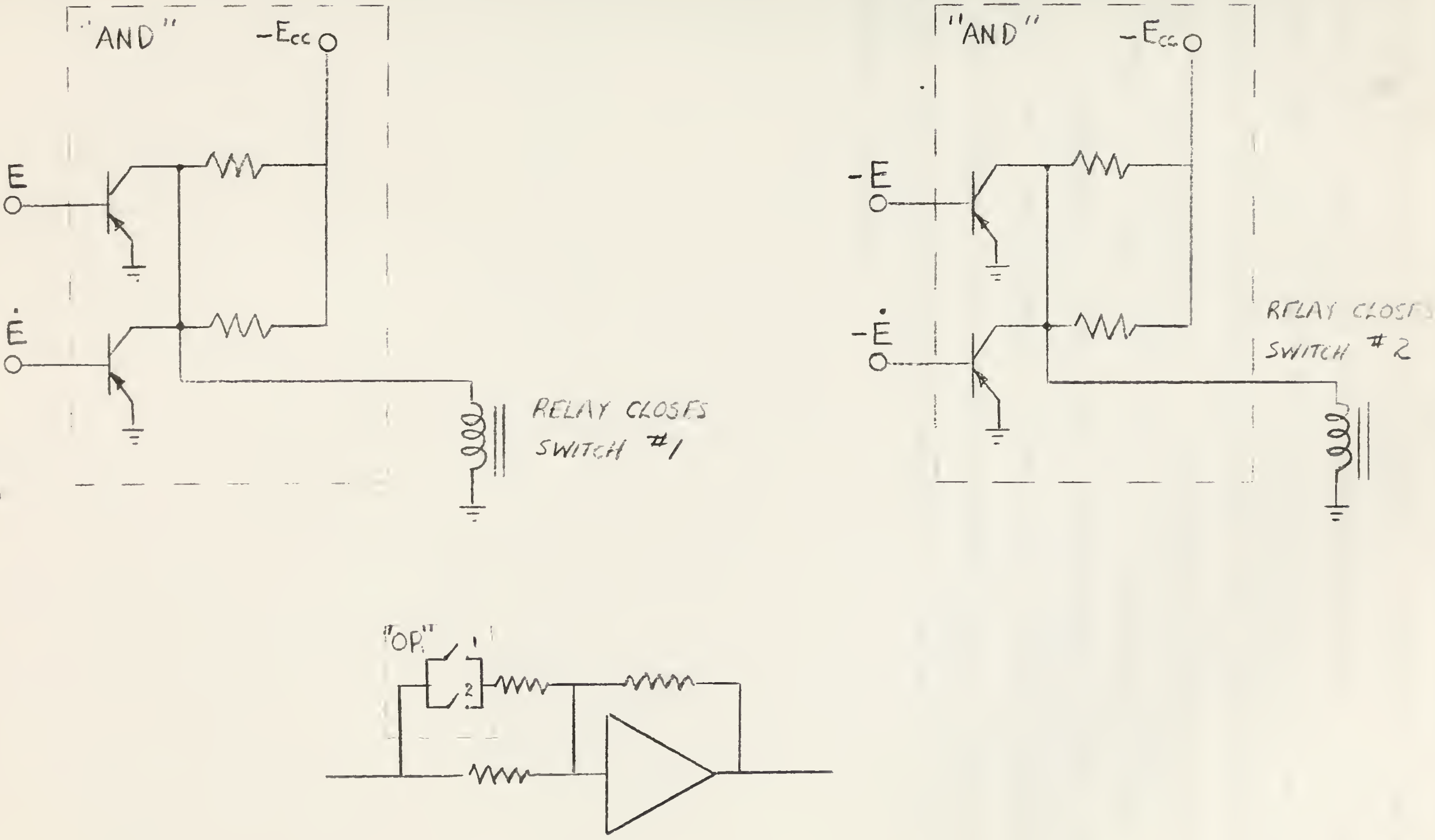


Fig. 5-2 Transistor "AND" Logic

amplifier. These auxiliary amplifiers are shown in Figure (5-3). The resistor in series with the input capacitor of amplifier A-3 depresses the noise pick up inherent with the differentiating circuit. Unfortunately the series resistor also introduces an undesired frequency and amplitude sensitive phase shift in the square wave output. For correct switching throughout a frequency response test it is necessary to adjust the capacitance value to compensate for phase shift each time the frequency is changed.

As discussed in a later chapter, midway through the step response tests it was found preferable to redefine the switching requirement in terms of rate of change of output position instead of rate of change of position error. Thus the differentiation amplifier is eliminated and the phase shift problem is circumvented.

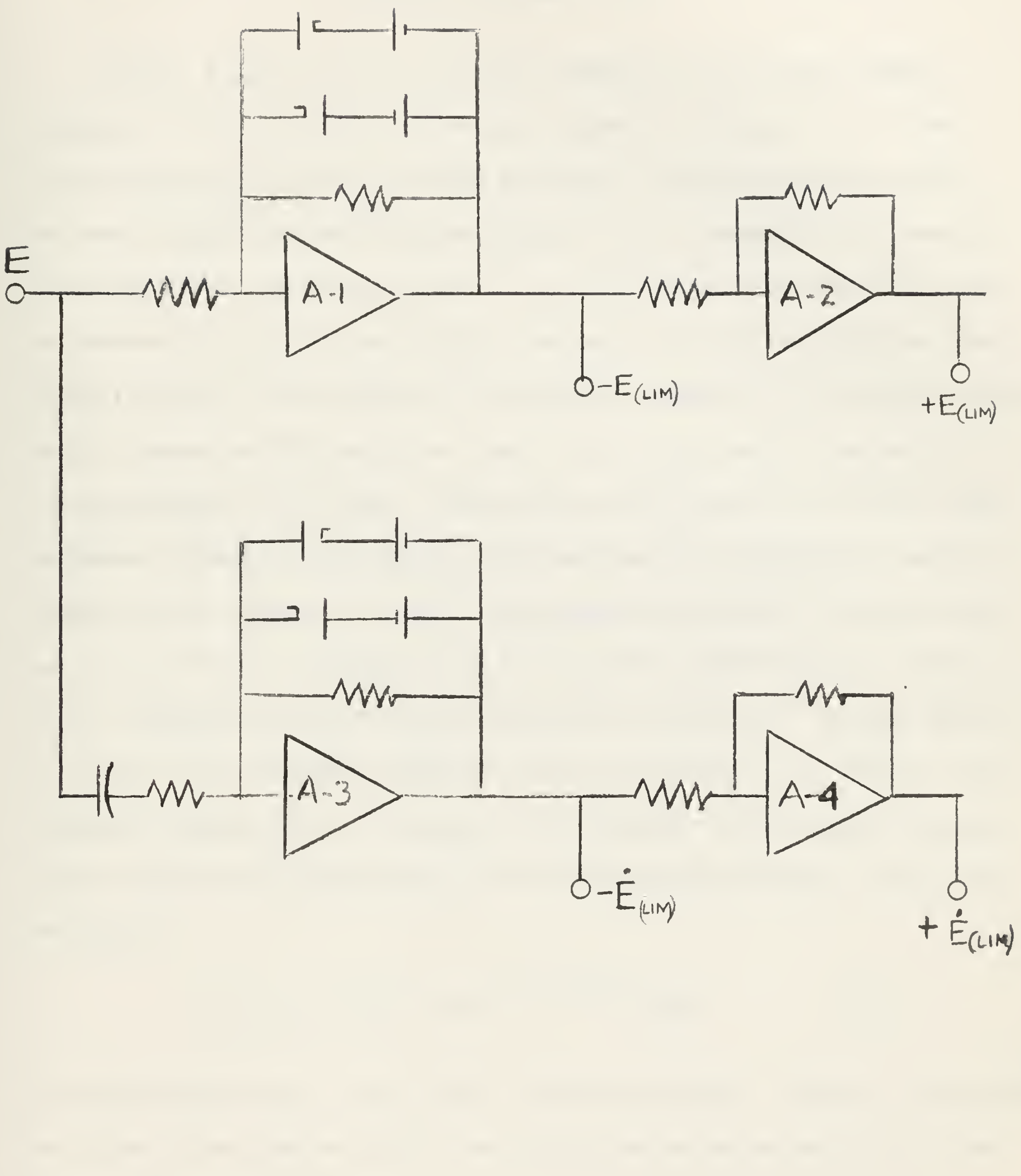


Fig. 5-3 Aux. Analog Circuit for Switching

CHAPTER VI

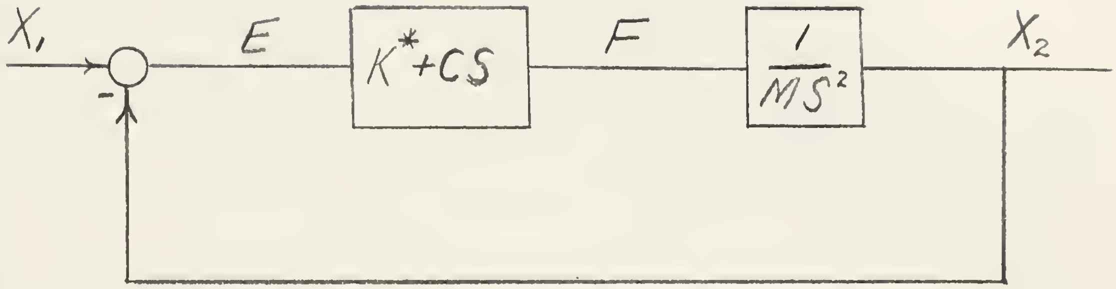
FREQUENCY RESPONSE TESTS

To have a basis for qualitatively comparing the two-rate spring suspension with a typical conventional suspension (Figure 3-1), closed loop frequency response tests were performed. The block diagram of the two-rate spring suspension system (Figure 6-1) is identical with that for the conventional suspension (Figure 2-1) except that the spring must now be represented with the describing function, G_d, or by the defining equations (4-1) and (4-2) instead of the constant numeric K. The larger spring-rate is chosen as 720 pounds per foot for all tests and M, one fourth of the sprung mass is 18 slugs. (These values are typical of a light weight passenger automobile although for this qualitative investigation numerical values are not critical. They do not change the character of the response but only serve as a scaling factor and here the interest is in a comparison of the responses with and without spring rate switching.) Viscous damping, C, is one of the variable parameters and for each test C is chosen to give a desired damping ratio. Throughout the remainder of the thesis "damping ratio" implies the damping ratio of the linear system without spring-rate switching.

$$\text{Damping Ratio} \equiv \zeta = C/2\sqrt{MK} \qquad (6\text{-}1)$$

B, spring-rate return, is the other variable parameter. Choosing B determines the value of the smaller spring rate since the larger spring rate is constant at 720 pounds per foot.

The Donner Model 3400 analog computer was used and Figure (6-2) shows the problem board circuit arrangement.



$K^* = 720$ lb/Ft when $E/\dot{E} > 0$

$= (720 - B)$ lb/Ft when $E/\dot{E} < 0$

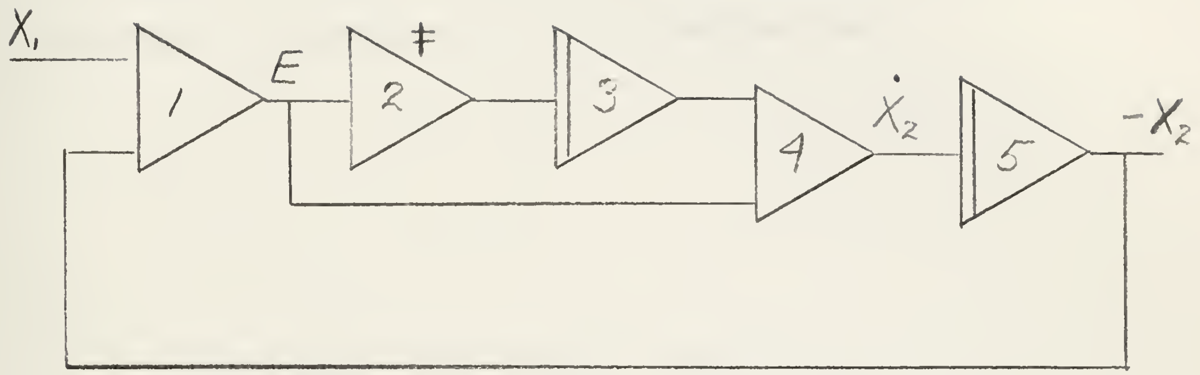
$M = 18$ slugs

C is variable

$\zeta = C/2\sqrt{MK} = C/227.8$

B is variable

Fig. 6-1 Block Diagram of the Two-rate Spring Suspension



‡ The auxiliary amplifiers and transistor-relay logic circuits required to switch the gain of amplifier 2 are shown in Figures (4-1), (4-2), and (4-3).

Fig. 6-2 Analog Computer Circuit for the Two-rate Spring Suspension System

To verify that the two-rate spring characteristic is not a function of input signal amplitude a preliminary test was made holding input frequency constant but varying input sine wave amplitude. This data, tabulated below, does indicate the spring characteristic to be independent of input signal amplitude. Therefore it is not necessary to hold position error constant throughout each frequency response test.

SINE WAVE AMPLITUDE OF INPUT TO TWO-RATE SPRING	X_1	X_2	X_2/X_1
9	1.86	3.75	2.01
19.2	3.98	8.02	2.01
30.7	6.37	12.60	1.98
41.0	8.75	17.50	2.00
54.7	11.90	23.3	1.96
65.5	13.80	27.5	1.99

Fig. 6-3 Constant Frequency Variable Amplitude Response

For the fast frequency response test no viscous damping was used (C = 0) and spring-rate return was set at B = 360 pounds per foot. This corresponds to 50 per cent spring return and zero damping ratio. The data are plotted on Figure (6-4.)

Without spring-rate return the amplitude of the resonance peak would be infinite. The finite resonance peak of Figure (6-4) demonstrates the damping effect at the spring rate switching. The natural frequency for the system is

$$\omega_n = \sqrt{K/M} = 6.32 \text{ RAD/SEC} \qquad (6\text{-}2)$$

For a lowly damped system the resonant frequency is very nearly equal to the natural frequency and therefore if spring rate switching were not applied to the system of Figure (6-4) then the resonance peak would occur at about 6.3 radians per second. Figure (6-4) indicates that with spring-rate switching a significant shift of the resonant frequency to the left is introduced. Recall that the comfort criteria presented in Chapter III dictates a reduction in the peak and a shift to the left to make resonance less perceptible to the passenger. Spring rate switching exhibits the ability to do both. However the resonance peak appears excessive for 50% spring return without viscous damping. Applying equation (3-6), for an arbitrary disturbance amplitude of two inches:

$$Af^3 = 2\left(\frac{5.5}{2\pi}\right)^3 \simeq 6.6$$

The jerk discomfort threshold limit is:

$$Af^3 = 2$$

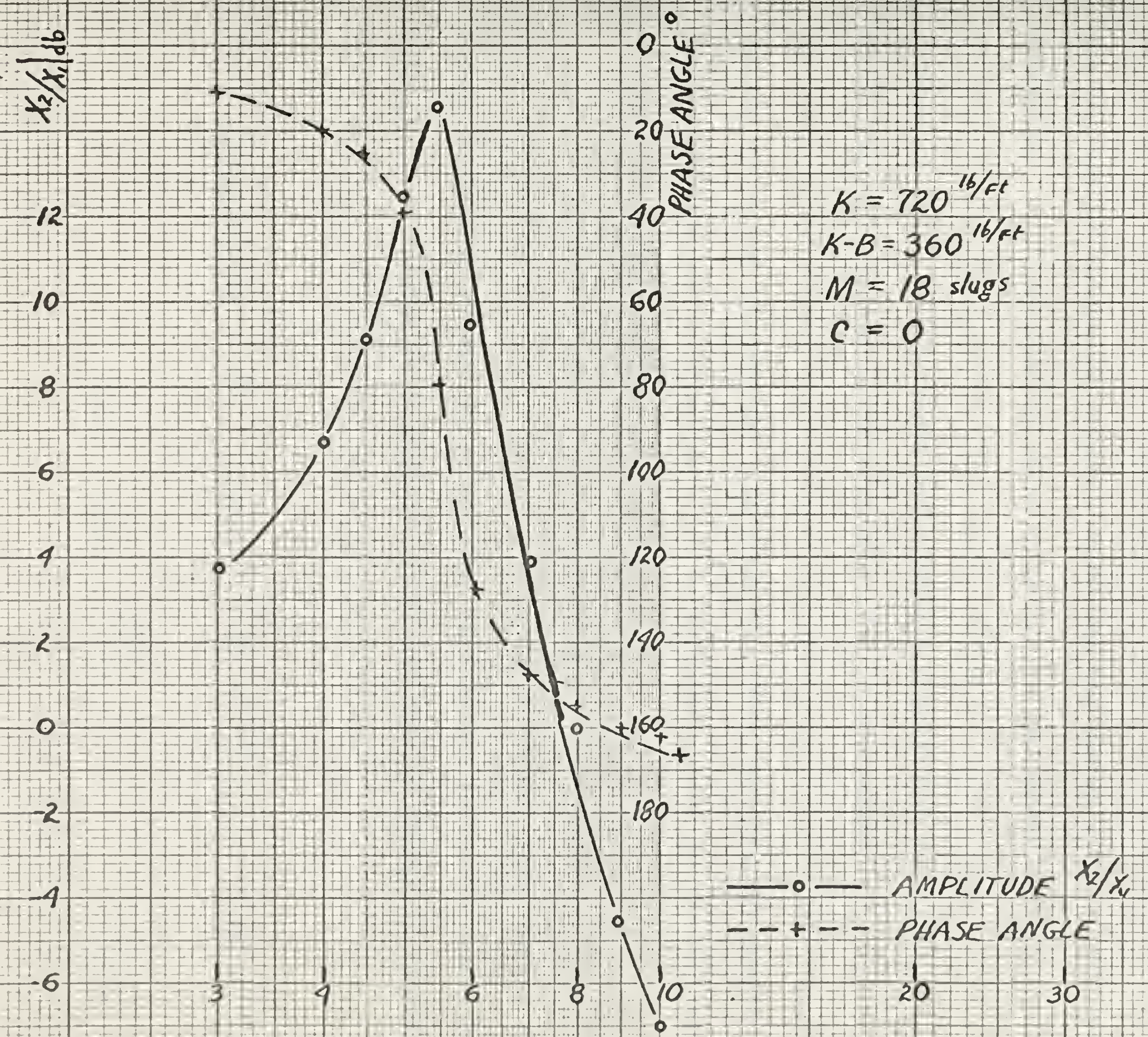


Fig 6-4 Frequency Response - No Viscous Damping

To reduce the resonance peak, greater spring return can be used or some viscous damping can be added. For the next series of frequency response tests, spring return, B, was held at 360 pounds per foot while tests were made with various amounts of viscous damping. Figures (6-5), (6-6), (6-7) and (6-8) display the frequency response data for damping ratios of 0.2, 0.3, 0.5, 0.7 respectively. Also plotted on each figure is the theoretical frequency response curve for the system without spring-rate switching. For each damping ratio value the spring-return resonance peak is smaller than the corresponding linear system resonance peak. This indicates that the spring-return contributes a significant amount of the total damping. Spring-return resonance peak is approximately two-thirds as great as the linear resonance peak for each case. For 50 per cent spring return, to reduce the maximum jerk below the limit specified by equation (3-6) a damping ratio larger than 0.2 is required, based on an arbitrary two inch disturbance amplitude.

Before proceeding with further frequency response testing, transient tests were performed. This data is presented in Chapter VII. During the transient testing it was discovered that spring-rate switching should be altered to be a function of position error and rate of change of output position, i.e., the modified defining equations are:

$$\text{Spring Rate} = K \qquad \text{when } E/\dot{X}_2 > 0 \tag{6-3}$$

$$= K - B \qquad \text{when } E/\dot{X}_2 < 0 \tag{6-4}$$

Several frequency response tests were performed with the modified switching. Figure (6-9) compares the original switching with the modified

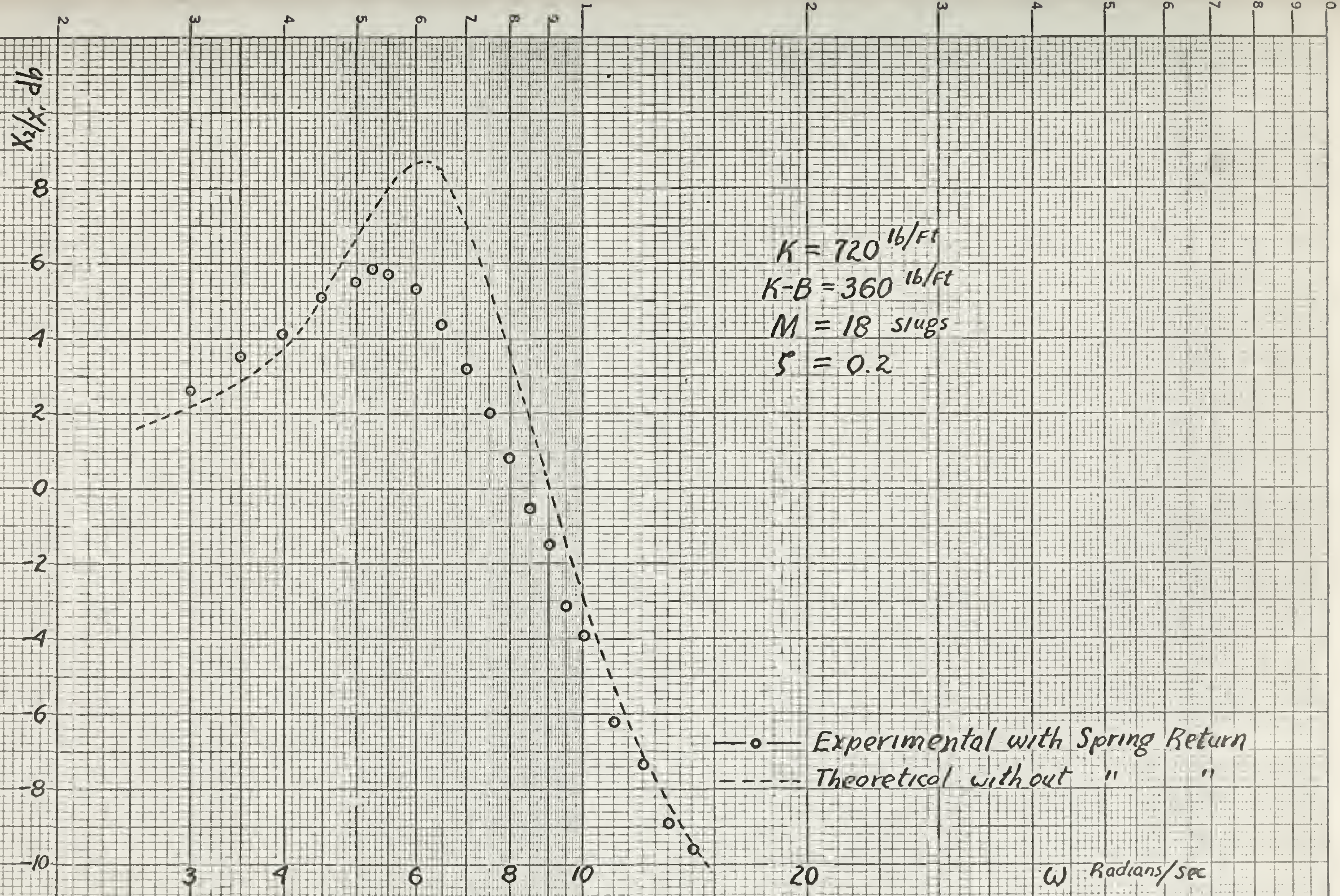


Fig. 6-5 Frequency Response - 50 % Spring Rate Return - 0.2 Damping Ratio

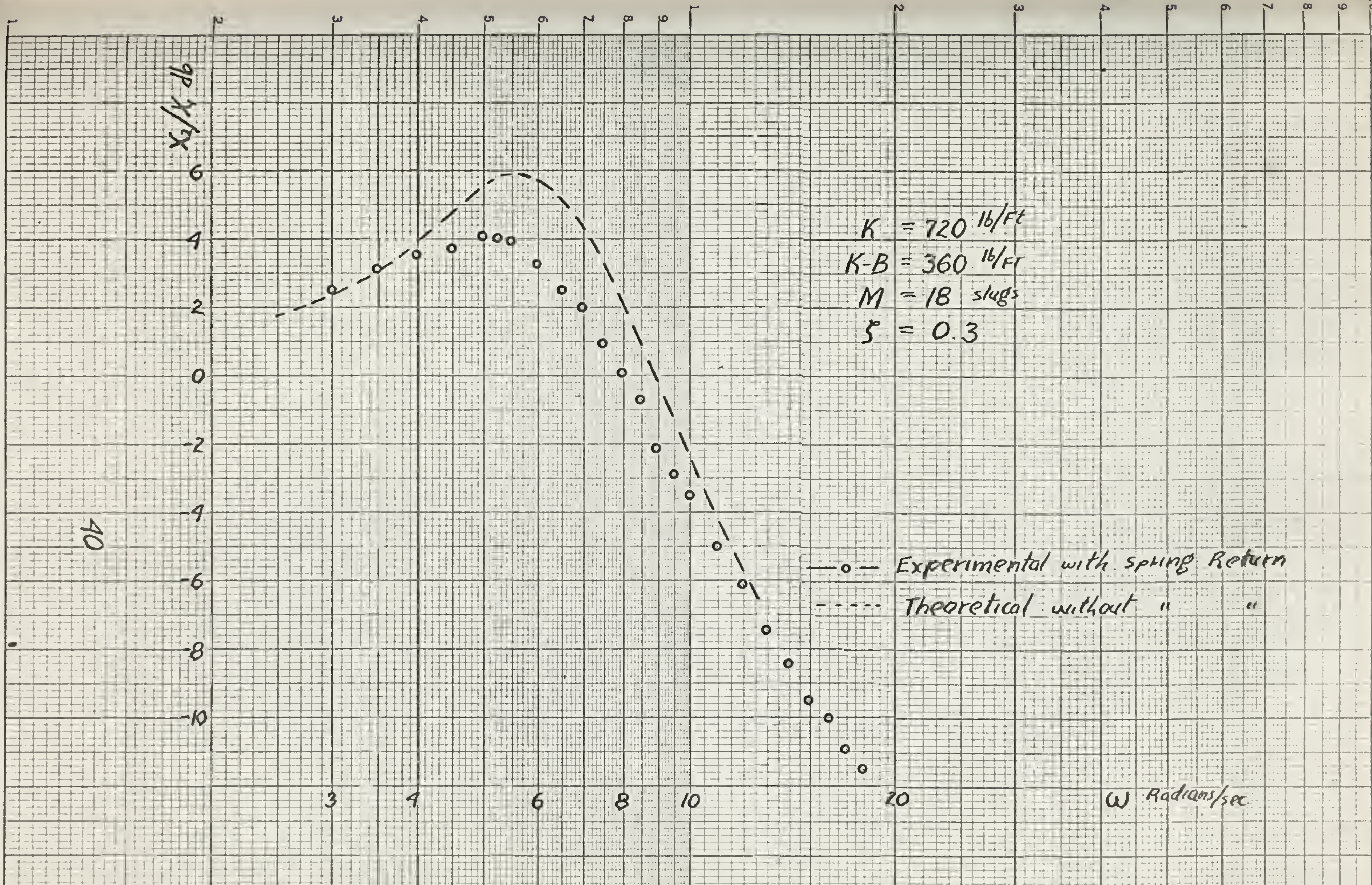


Fig. 6-6 Frequency Response - 50 % Spring Return - 0.3 Damping Ratio

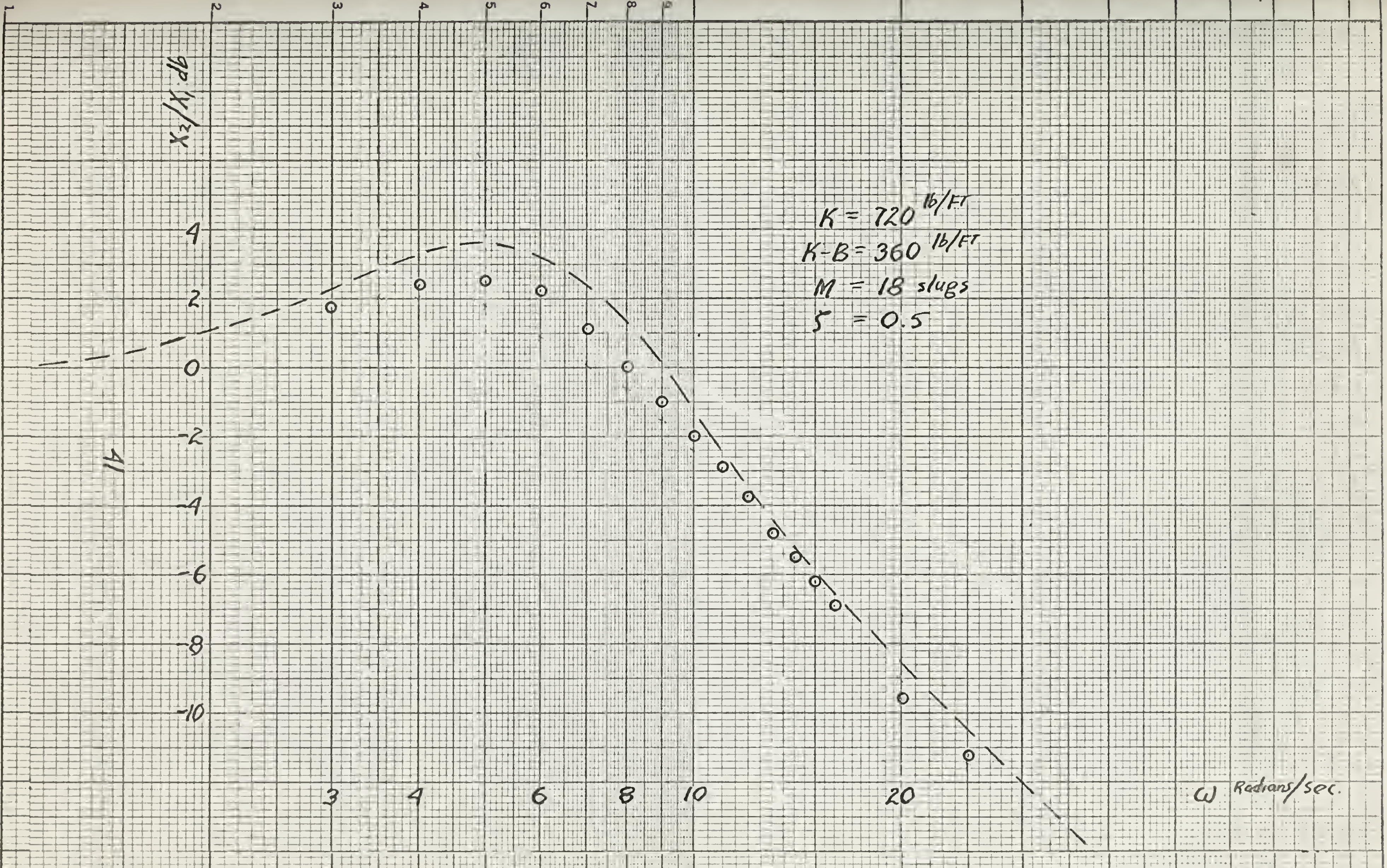


Fig. 6-7 Frequency Response - 50% Spring Return - 0.5 Damping Ratio

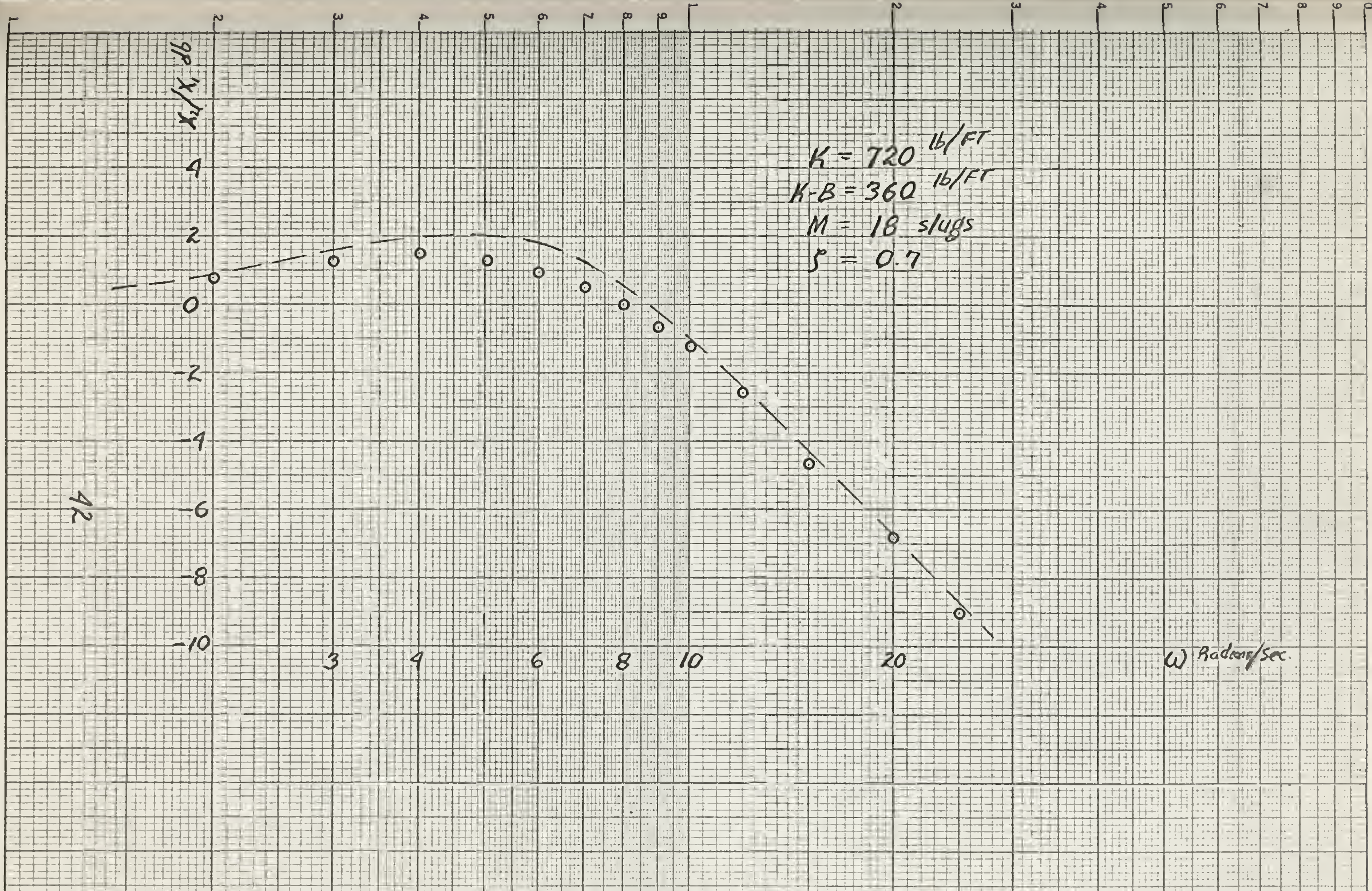


Fig 6-8 Frequency Response - 50% Spring Return - 0.7 Damping Ratio

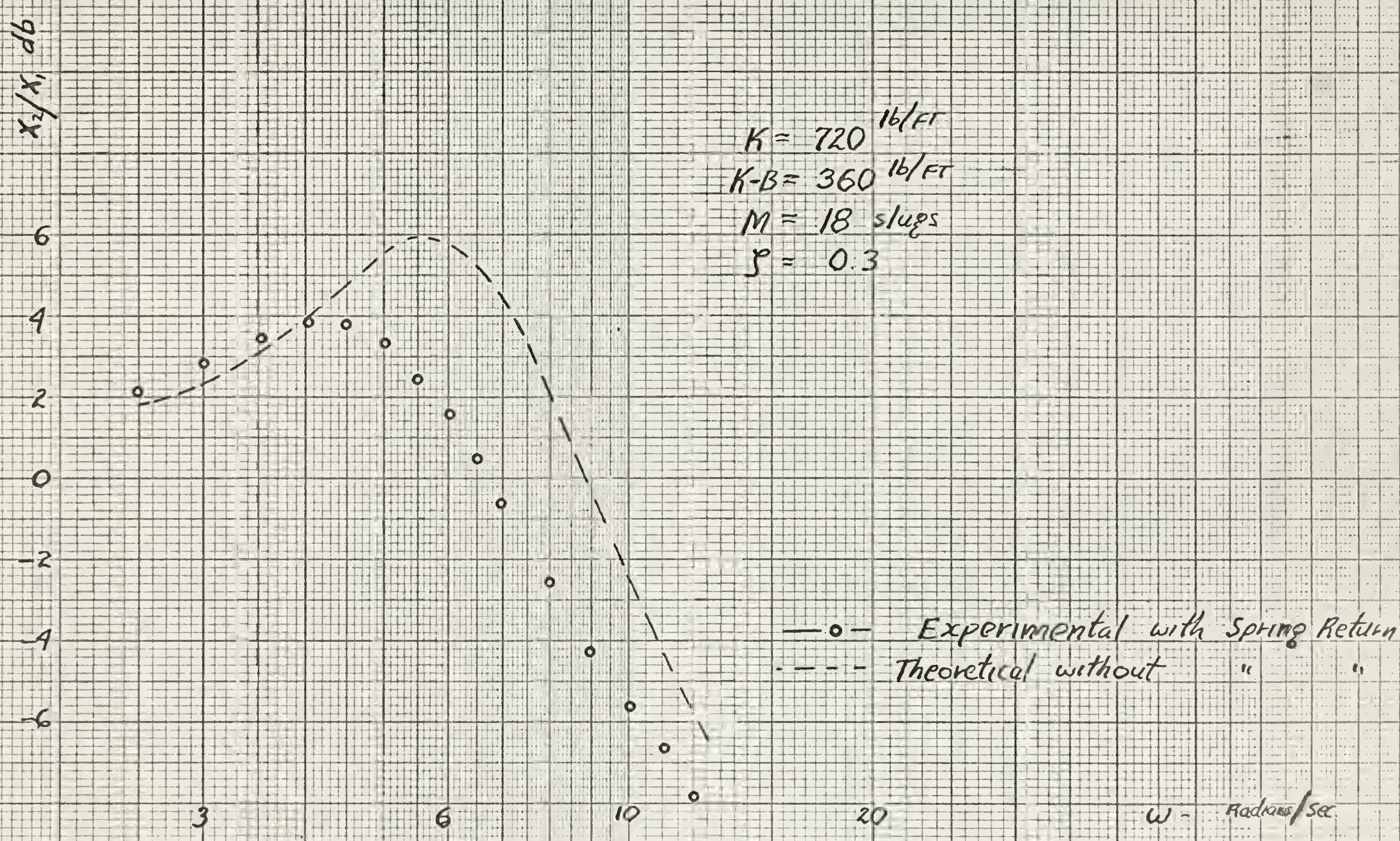


Fig 6-9 Frequency Response - Modified Switching - 50% Return - 0.3 Damping Ratio

switching for 50 per cent spring return and 0.3 damping ratio. With modified switching a smaller resonance peak and a greater resonant frequency shift is evident.

For the final frequency response tests modified switching was incorporated with greater spring-return. Figure (6-10) presents the results for 75% spring-return and 0.3 damping ratio, and Figure (6-11) gives the response for 90% spring-return and 0.1 damping ratio. As expected greater spring return provides greater damping and shifts the resonant peak further to the left. Considering only frequency response data the characteristic of Figure (6-11) is better than for all other parameter combinations investigated. Adequate damping is provided to meet the jerk comfort criteria with only a small amount of viscous damping.

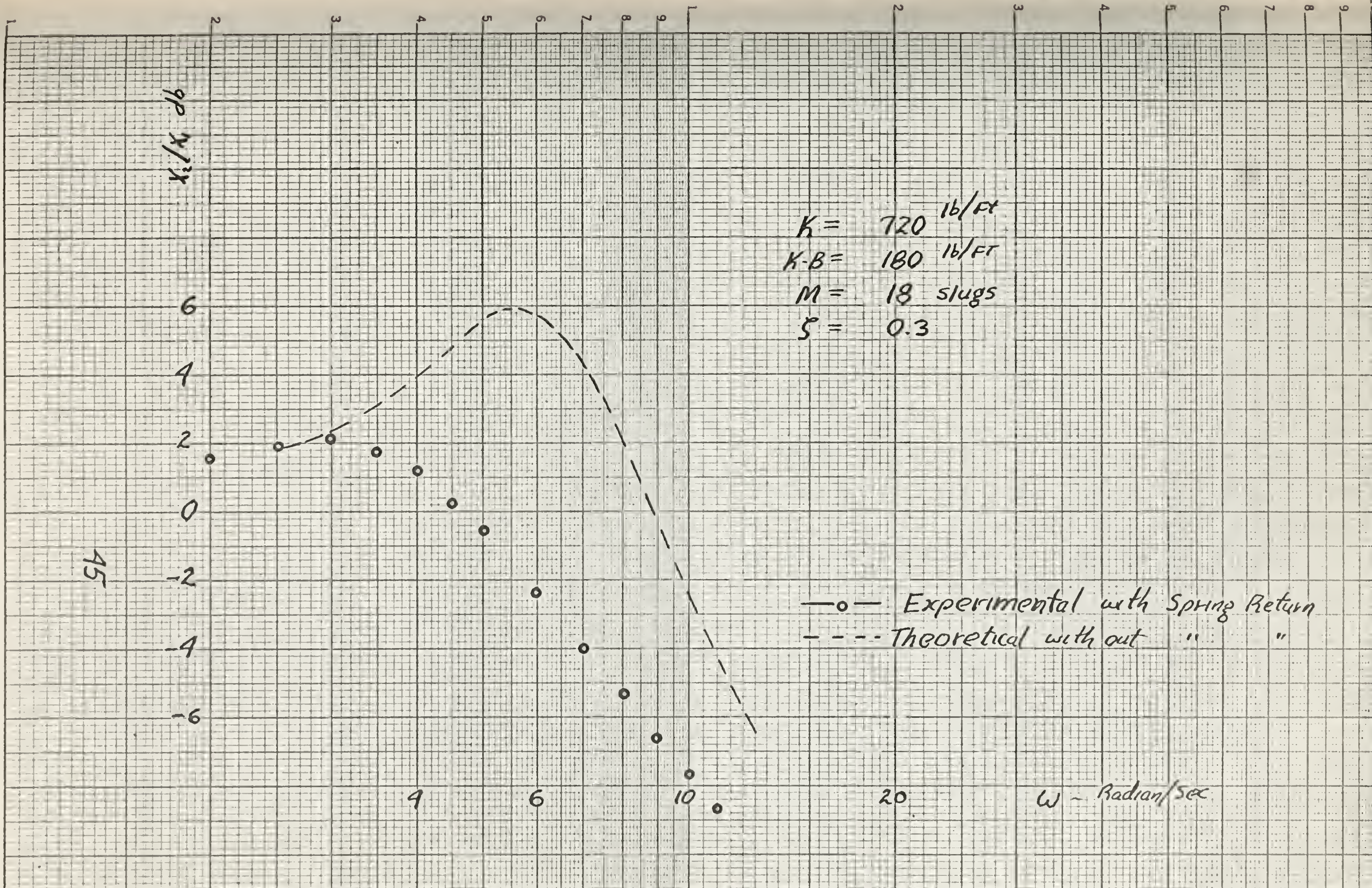


Fig. 6-10 Freq Response - Modified Switching - 75 % Return - 0.3 Damping Ratio

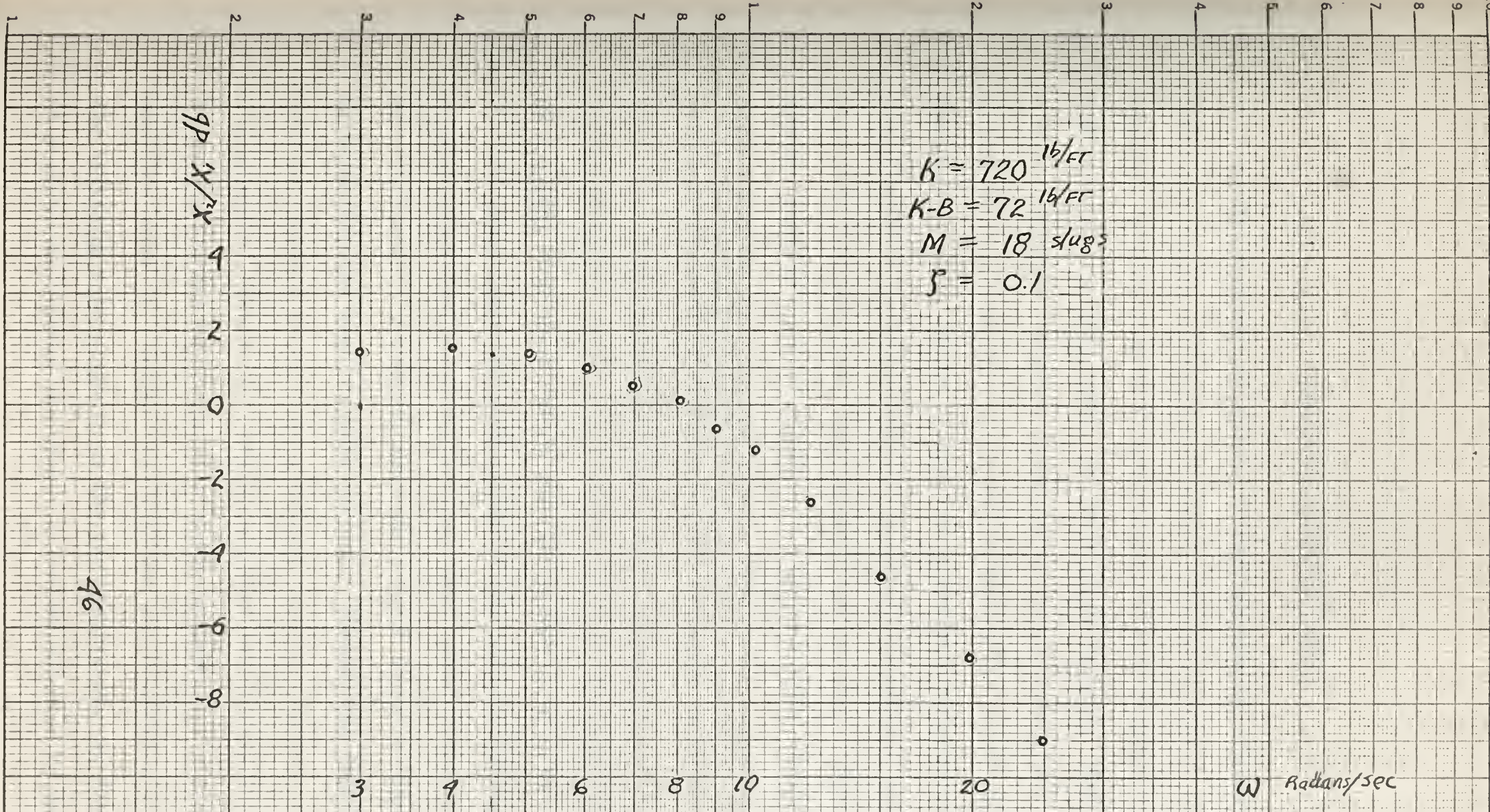


Fig 6-11 Freq. Response - 90% Return - with Modified Switching - 0.1 Damping Ratio

CHAPTER VII

TRANSIENT RESPONSE TESTS

Servomechanisms are rarely if ever excited by sinusoidal signals. Therefore frequency response characteristics are useful to the control engineer only as a means of predicting the time response of a servomechanism subjected to step, ramp, or other input signals. When possible the time or transient response characteristics are determined directly by applying an input such as a step and observing the output quantity as a function of time. So it is with the automobile suspension system. Washboard surfaces do approximate a sinusoidal disturbance but the automobile must ride over aperiodic road imperfections as well. Note that the frequency response curve of Figure (6-8) where a large damping ratio is employed does not disclose the adverse derivative characteristic associated with suddenly applied disturbances and discussed in Chapter II. Recall that it is this characteristic of transmitting high accelerating forces to the passenger when a wheel hits a pot-hole that precludes the design of a <u>linear</u> suspension system that is optimum in all ride characteristics.

To investigate the performance of the two-rate spring when the automobile passes over pot-holes or rides over curbs, transient response tests were performed. The analog computer circuit shown in Figure (6-2) is also satisfactory for step response testing. However the logic switching circuit does not perform as well for transient testing. Whereas the differentiating capacitor (Figure 5-3) can be adjusted for correct switching before a frequency run and switching will remain correct, for a step response run position error is a damped sine wave and the decreasing amplitude advances the switching time slightly as the run progresses. To minimize the effects of switch time advance the capacitor can be adjusted to cause slightly late

switching at the start of the response to compensate for advance during the run.

A common method of displaying the transient response characteristics is on plots of output position vs. time, position error vs. time, output velocity vs. time, etc. However, the effect of spring-rate switching on the response of the suspension to a step input can better be analyzed on a plot of position error vs. rate of change of position error. Such a plot is known as a phase trajectory and a family of several phase trajectories, each for a different step magnitude, forms a phase portrait. Interpretation of the experimental data is enhanced by a preliminary discussion of the expected shape of the phase trajectory.

From the initial value theorem the values of position error and rate of change of position error immediately following application of a step are:*

$$E(0) = A \qquad (7\text{-}1)$$

$$\dot{E}(0) = (C/M)A \qquad (7\text{-}2)$$

On the sketch of Figure (7-1) the above initial values are the coordinates of point (0). The trajectory starts at the origin before the step input and arrives at point (0) immediately after; the path in between is undefined. For the segment between (0) and (1) the small spring-rate is effective and the body swings back toward equilibrium. Note the gradual curvature indicating low acceleration. Passing through (1) the position error reverses sign and the large spring-rate is switched in as the body overshoots the equilibrium position. The greater curvature in the third quadrant indicates greater deceleration resulting from the large spring-rate. $\dot{E}$

*The initial value calculation appears in Appendix II

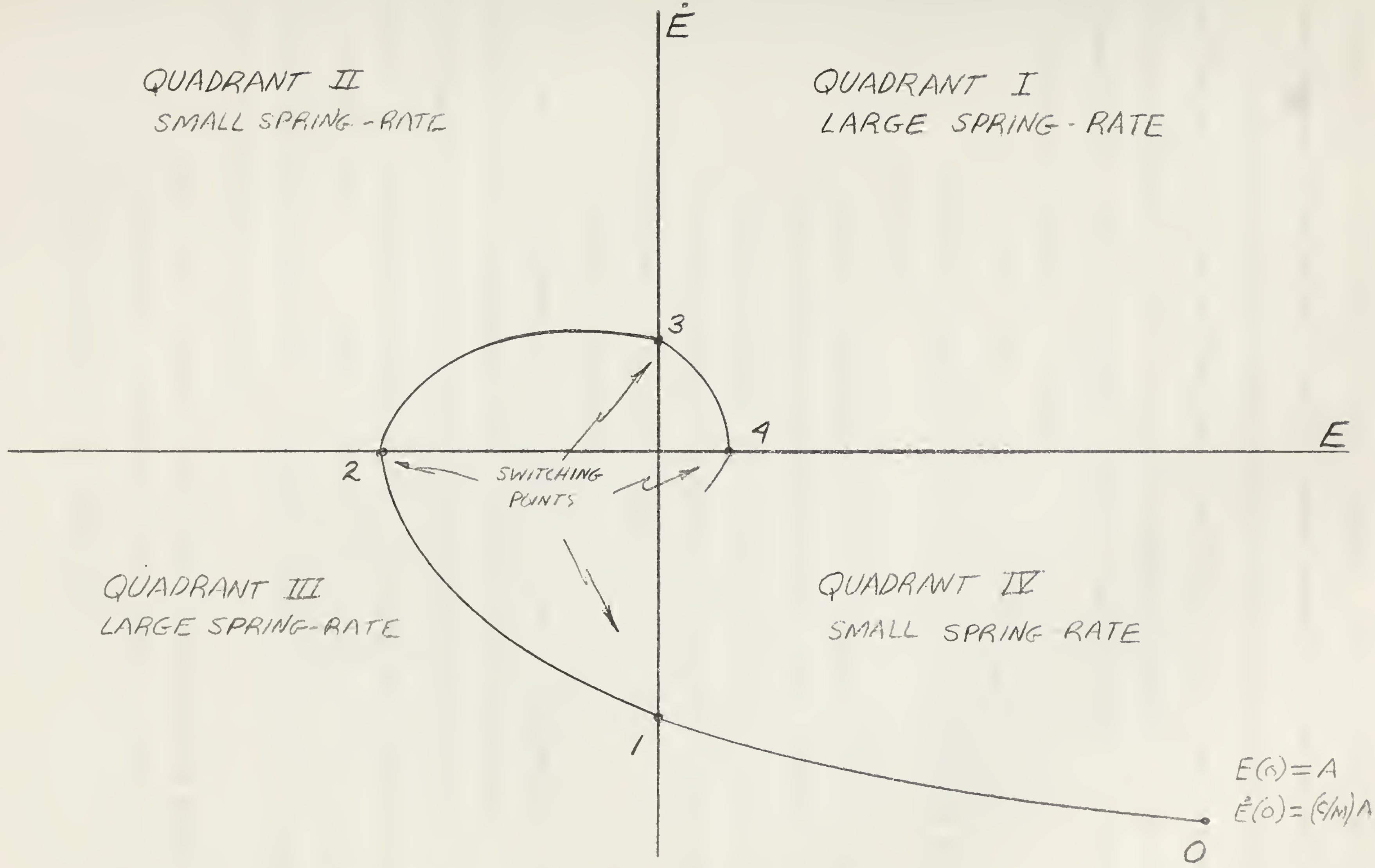


Fig. 7-1 Typical Phase Portrait for the Two-rate Spring Suspension

changes sign as the trajectory enters quadrant II and the small spring rate becomes effective while the body moves toward equilibrium position.

For the first series of step response tests, switching is determined by Thompson's original defining relations (Figure 2-2). The phase portraits of Figure (7-2) and (7-3) are for damping ratios of 0.3 and 0.5 respectively. 50 per cent spring return is used for both. An analysis of Figures (7-2) and (7-3) follows:

1. For equal step inputs the larger damping ratio of (7-3) produces a greater initial velocity as predicted by Equation (7-2).
2. Curvature in quadrants IV and III is greater for (7-3) demonstrating larger decelerating damper force. Although initial velocity is greater in (7-3) the velocities of (7-2) and (7-3) have equalized at the end of the fourth quadrant segment due to the larger damping ratio in (7-3).
3. Overshoot is less for (7-3).

None of the trajectories of Figures (7-2) and (7-3) display a discontinuous slope at switching points. The next step response test is conducted with more spring return to make switching discontinuities apparent. Figure (7-4) presents the phase portrait for 75 per cent spring return and 0.3 damping ratio. The effect of the increased spring return can be seen by comparing with Figure (7-2), the phase portrait for the same damping ratio but for 50 per cent return.

1. For equal step inputs the initial velocities are equal. This is consistent with Equation (7-2) which is a function only of viscous damping and the sprung mass.

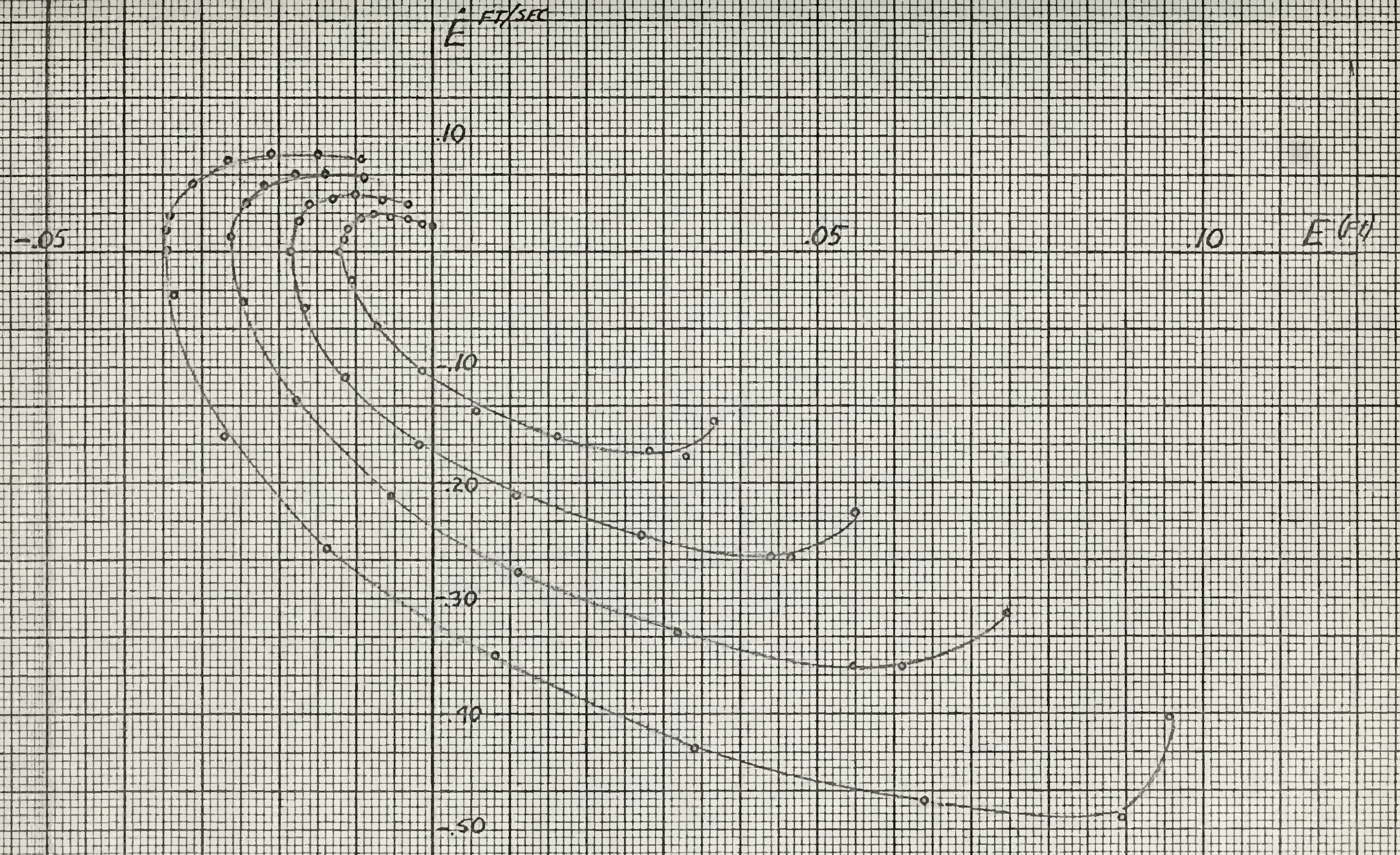


Fig 7-2 Phase Portrait - 50% Spring Return - 0.3 Damping Ratio

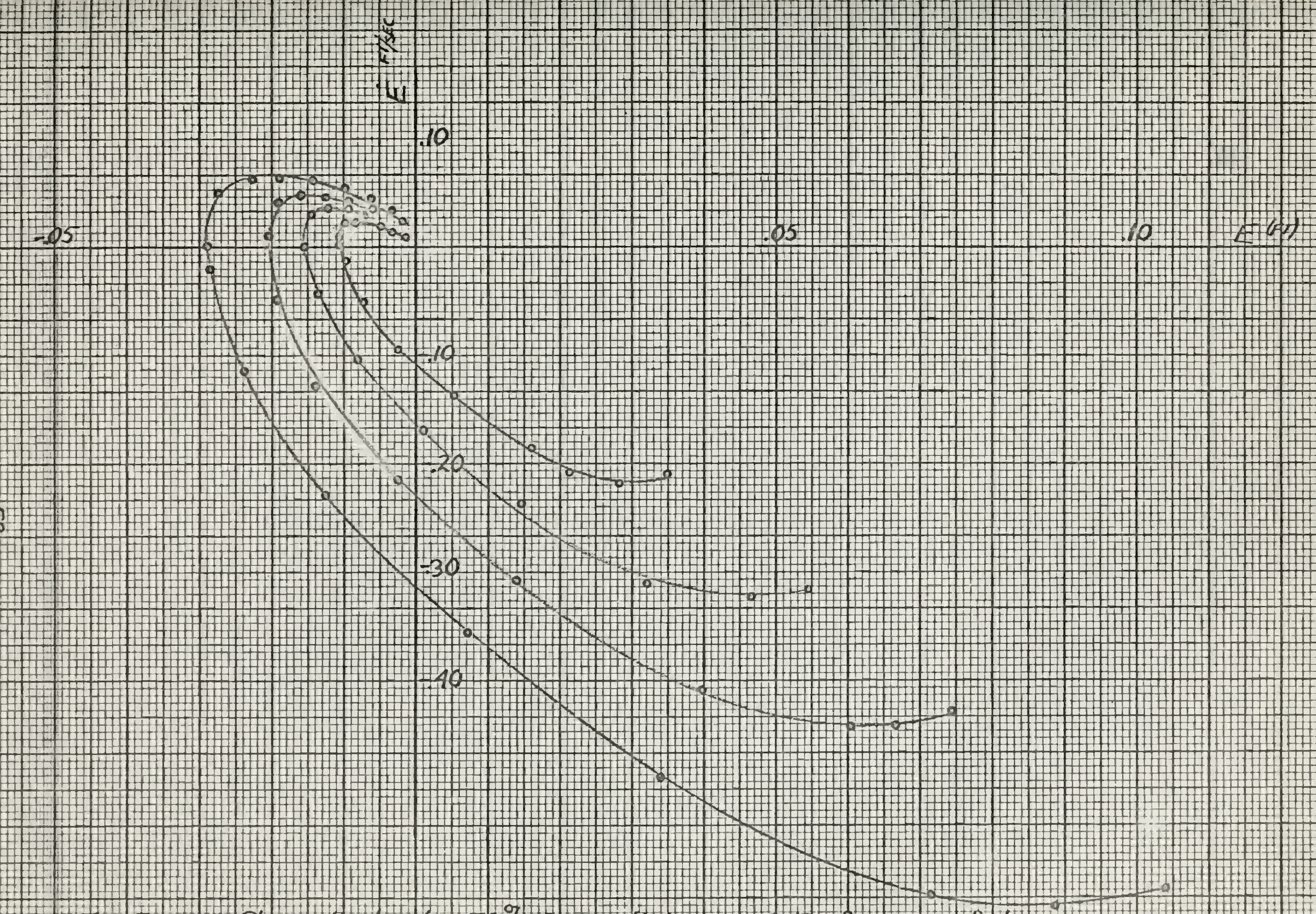


Fig 7-3 Phase Portrait - 50% Spring Return - 0.5 Damping Ratio

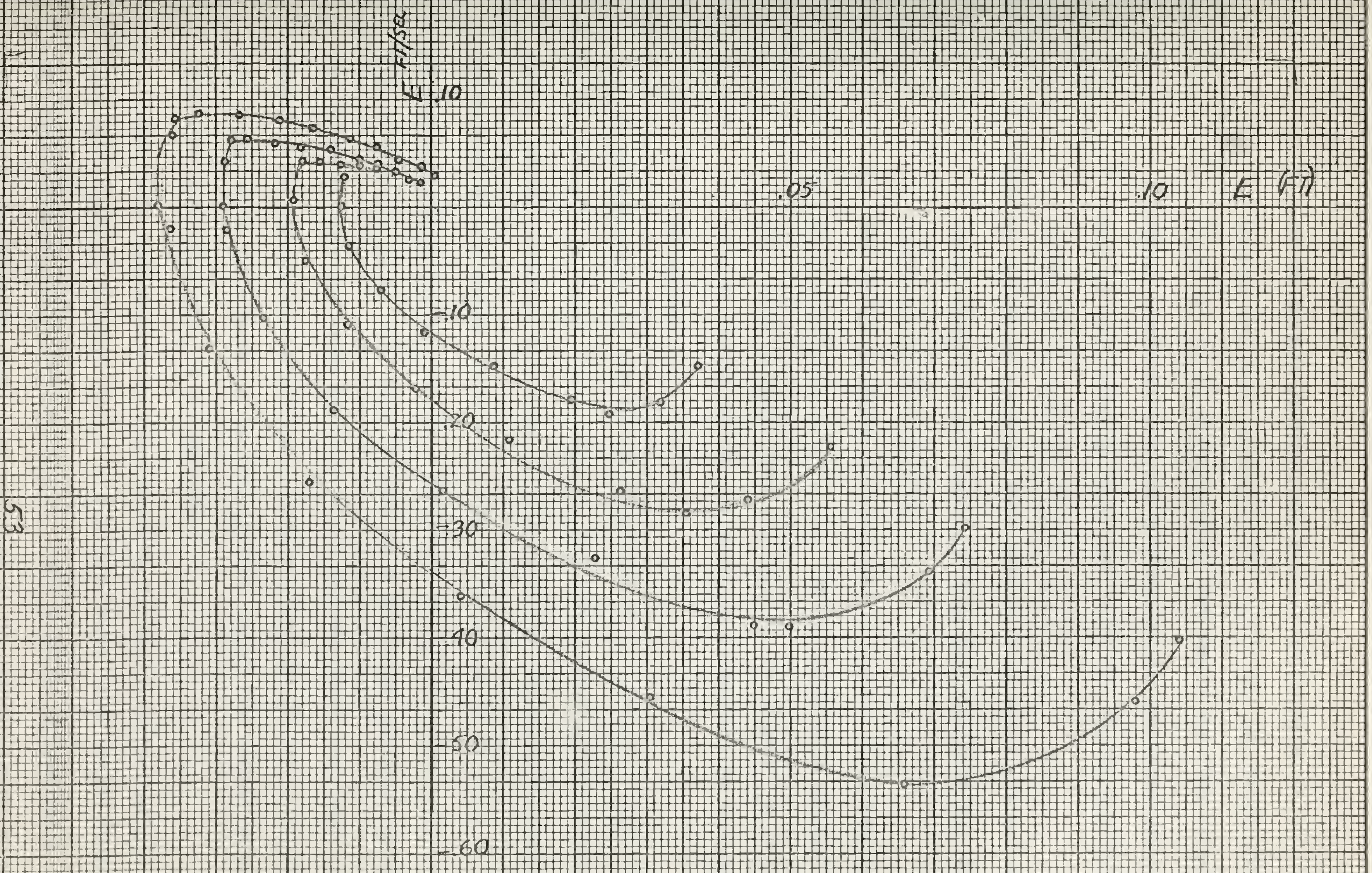


Fig. 7-4 Phase Portrait - 75% Spring Return - 0.3 Damping Ratio

53

2. In the fourth quadrant the smaller spring-rate is effective for each portrait; 360 lbs/Ft and 180 lbs/Ft for (7-2) and (7-4) respectively. For equal position errors (spring compressions) the smaller spring rate develops a smaller decelerating force and hence the fourth quadrant curvature is less for (7-4).

3. In the third quadrant both systems have equal (large) spring-rates effective and therefore both families have the same curvature in this quadrant.

4. The second quadrant trajectories of Figure (7-4) exhibit marked discontinuities of slope. It is noted with concern that this discontinuity does not fall on the negative E coordinate axis as called for by the defining equations for spring-rate switching.

An investigation of the output waveforms of all computer amplifiers in the circuit of Figure (6-2) explains the late switching encountered above. These waveforms are shown in Figure (7-5). Harmonic distortion introduced in the spring force wave form by switching is not well filtered by the succeeding circuit stages. Therefore distorted output position subtracts from the input sine wave to produce a distorted position error. When the slope of this distorted position error becomes zero, the logic circuit performs the spring-rate switching. However, because of the distortion and the phase shift introduced by the spring-rate return the output velocity is past zero when switching is accomplished.

The above explanation of the switching difficulty also suggests a solution: Replace rate of change of position error with output velocity for logic circuit switch determination. This solution conveniently eliminates the differentiating amplifier and with it the necessity to continually adjust for correct switching. In the remainder of the thesis switching

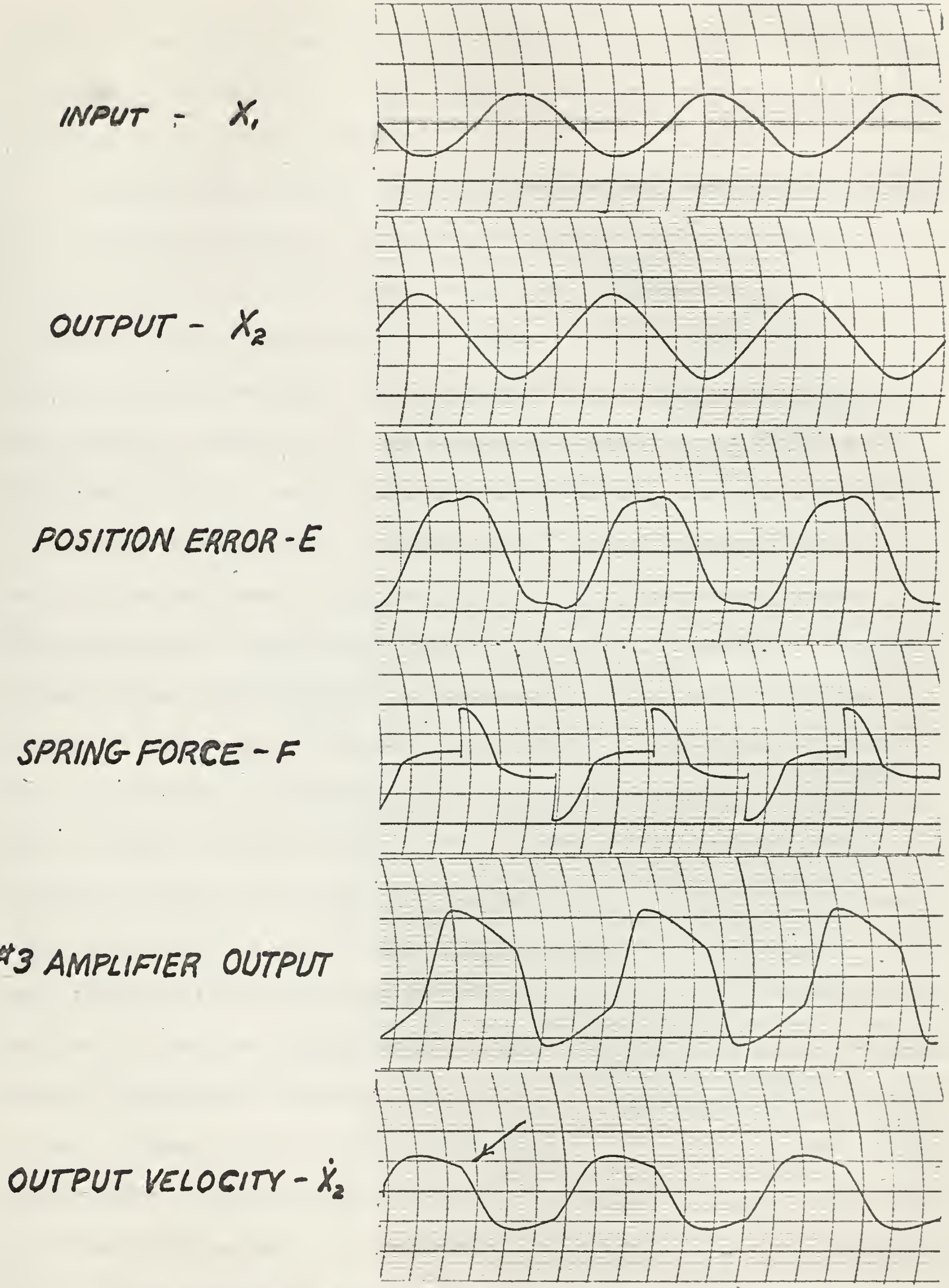


Fig. 7-5. Analog Computer Wave Forms

based on Thompson's proposal will be referred to as original switching. Switching as a function of output velocity and position error will be referred to as modified switching.

A final series of phase portraits were obtained using modified switching. Figures (7-6) and (7-7) present the results for damping ratios of 0.3 and 0.5 both with 50% spring-return. The increased damping effect obtained with modified switching is apparent in (7-6) and (7-7). A physical explanation can be given for the reduced damping that results from original switching. As the trajectory approaches the negative E coordinate axis the large spring-rate is effective. Maximum compression is attained at cross-over, and because the velocity instantaneously is zero all of the system energy is contained in the compressed spring. If the spring rate is switched at this point some of this energy is dissipated and spring force decreases instantaneously to the small spring-rate value. If switching is late the large spring force associated with the fully compressed large spring-rate is available to accelerate the sprung mass. Higher velocity results and when switching finally occurs much of the potential energy of the spring has already been transferred to the sprung mass and therefore a smaller amount is dissipated by the switching. Figures (7-8) and (7-9) show phase trajectories for 0.1 and 0.3 damping ratios, both with 75 per cent spring-return. Figure (7-8) exhibits a desirable low initial velocity but overshoot is excessive and more return damping is required. Figures (7-10) and (7-11) present the final tests. Damping ratios are 0.05 and 0.1 respectively and spring-rate return is 90 per cent. Both desirable characteristics are exhibited. Initial velocity is low because of small viscous damping but spring-rate return damping is adequate to limit the overshoot.

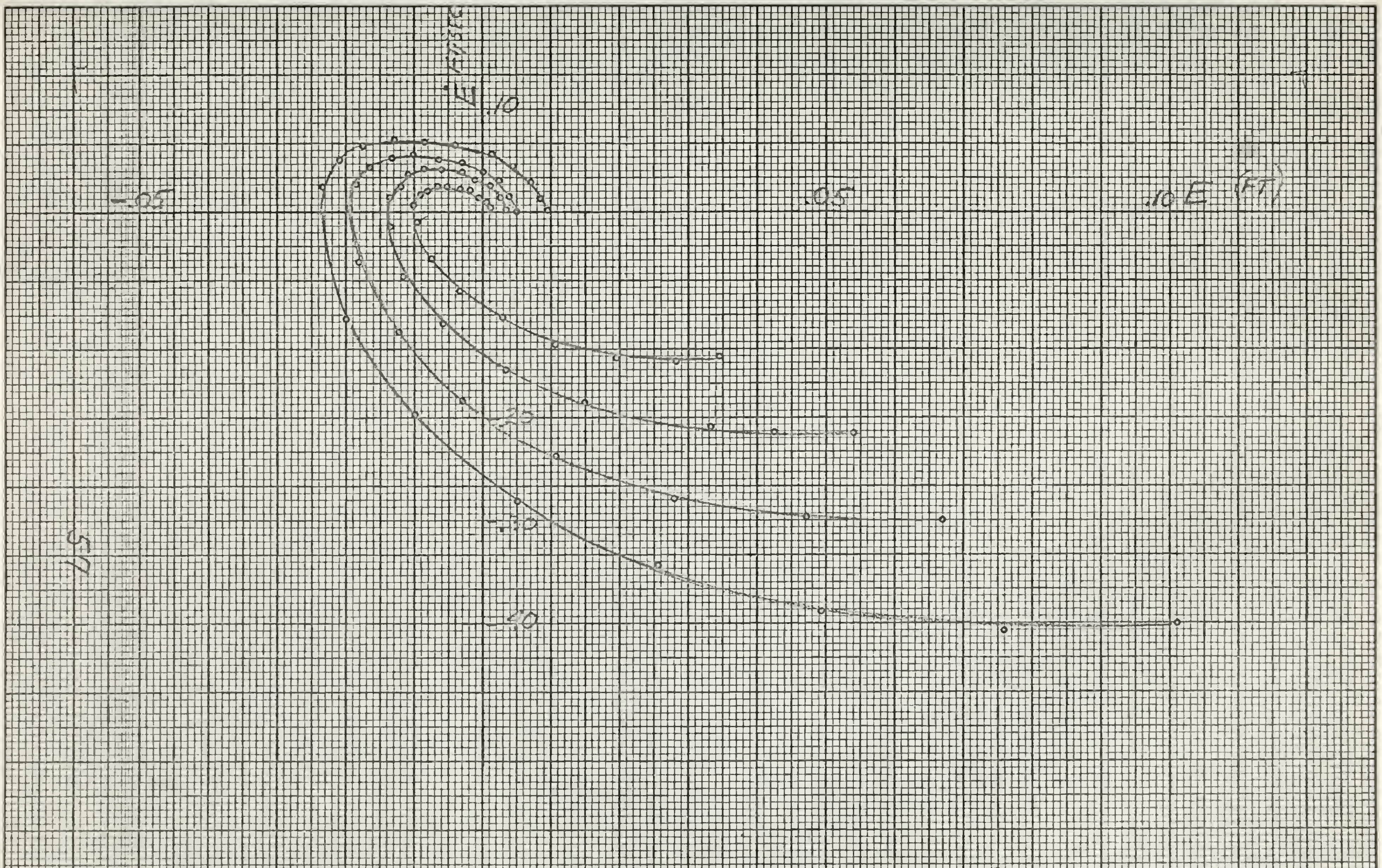


Fig 7-6 Phase Portrait with Modified Switching - 50% Return - $\rho = 0.3$

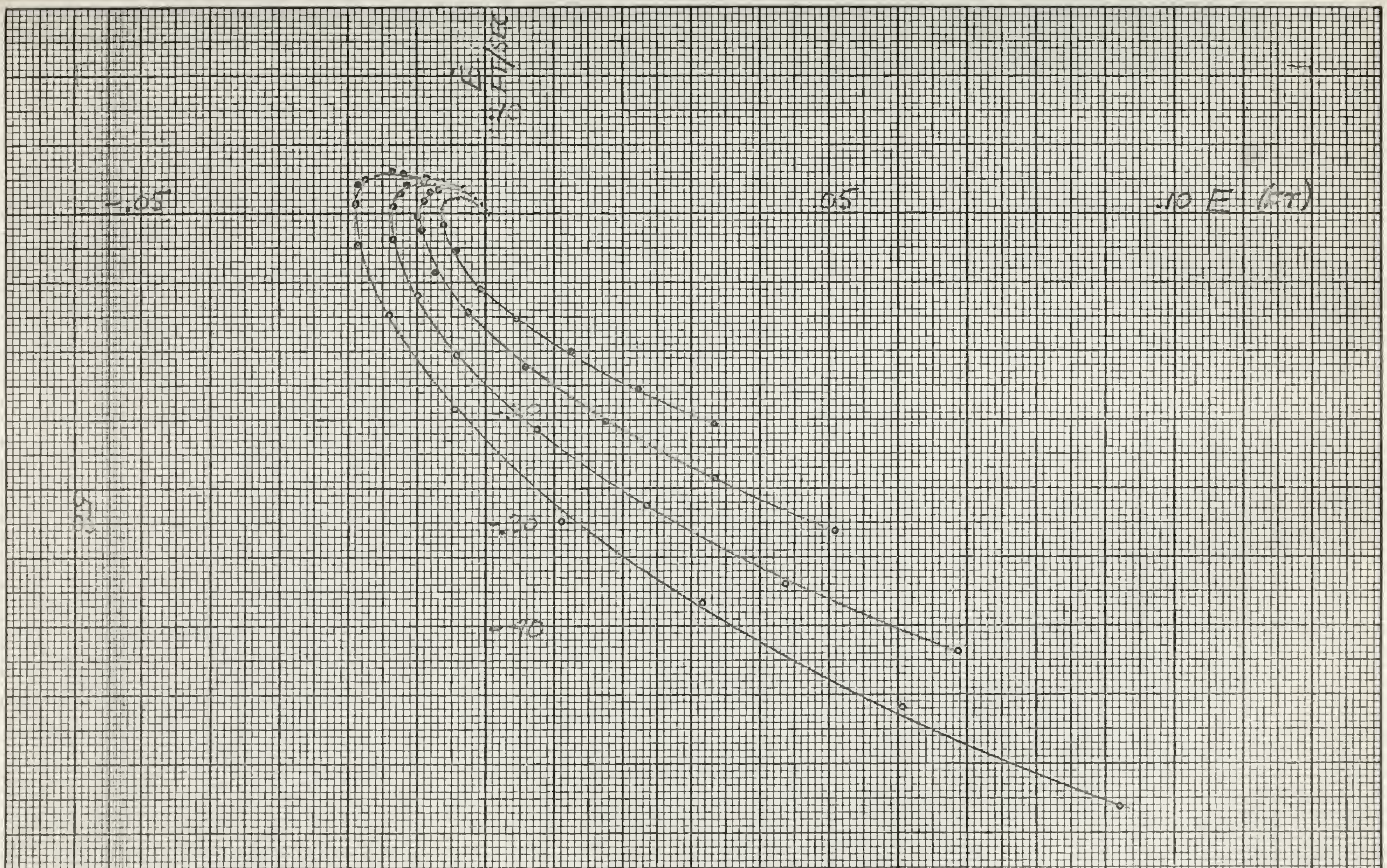


Fig. 7-7 Phase Portrait - with Mod. Switching - 50% Return - ζ = 0.5

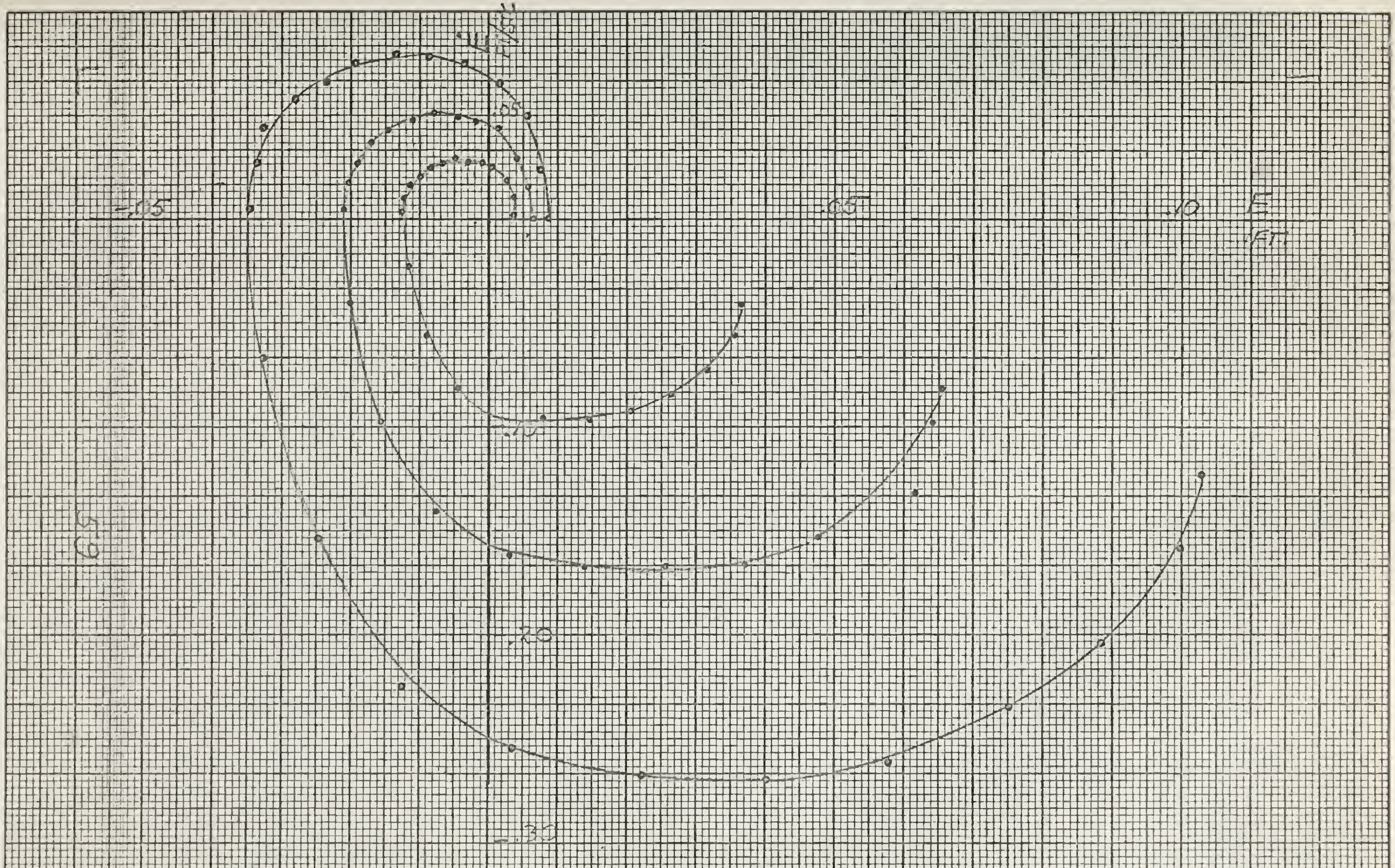


Fig 7-8 Phase Portrait - Mod. Switching 75% Return - 0.1 Damping Ratio

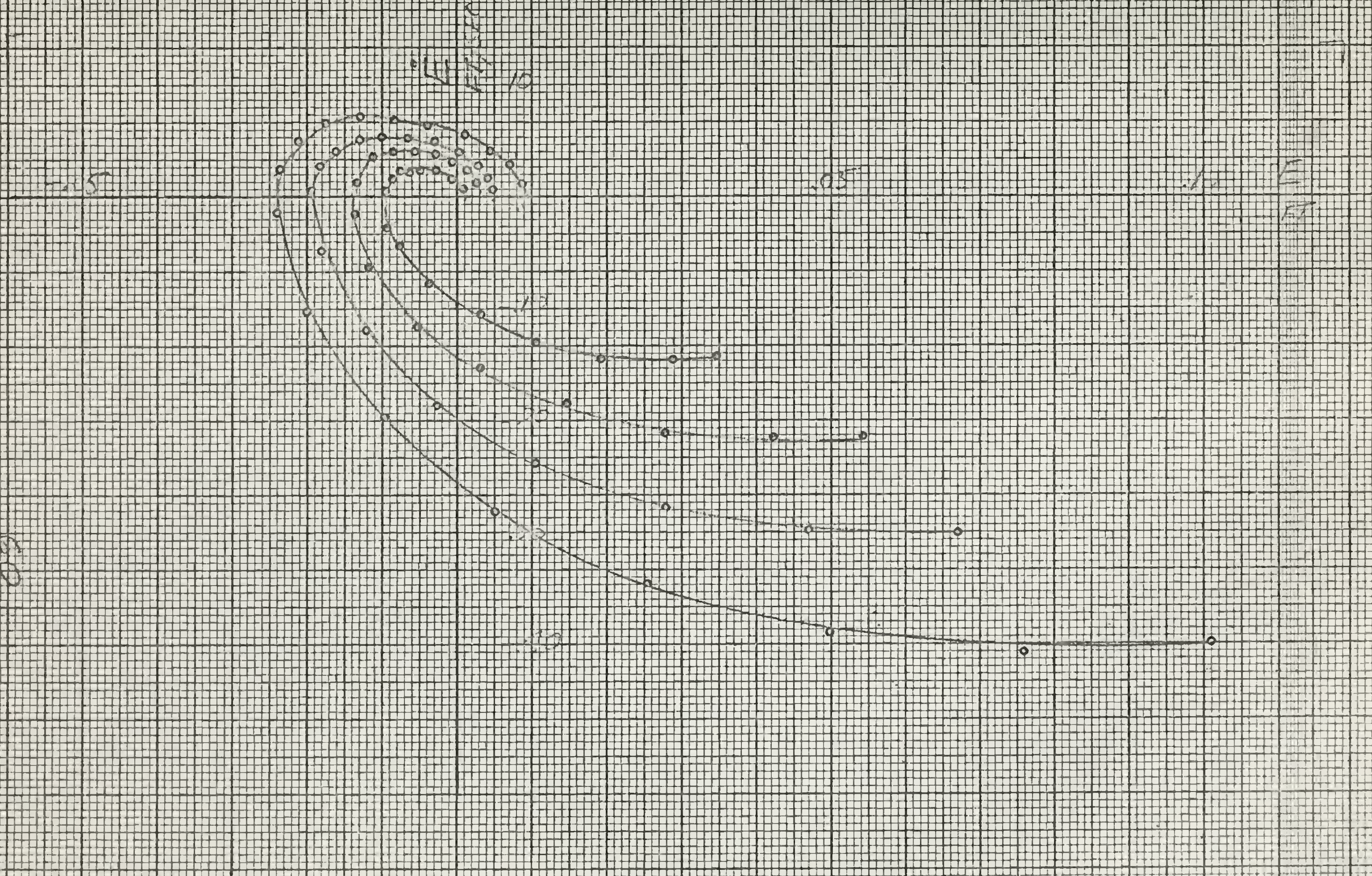


Fig 7-13 Phase Portrait with Max Switching + Step Return - 0.3 Damping Ratio

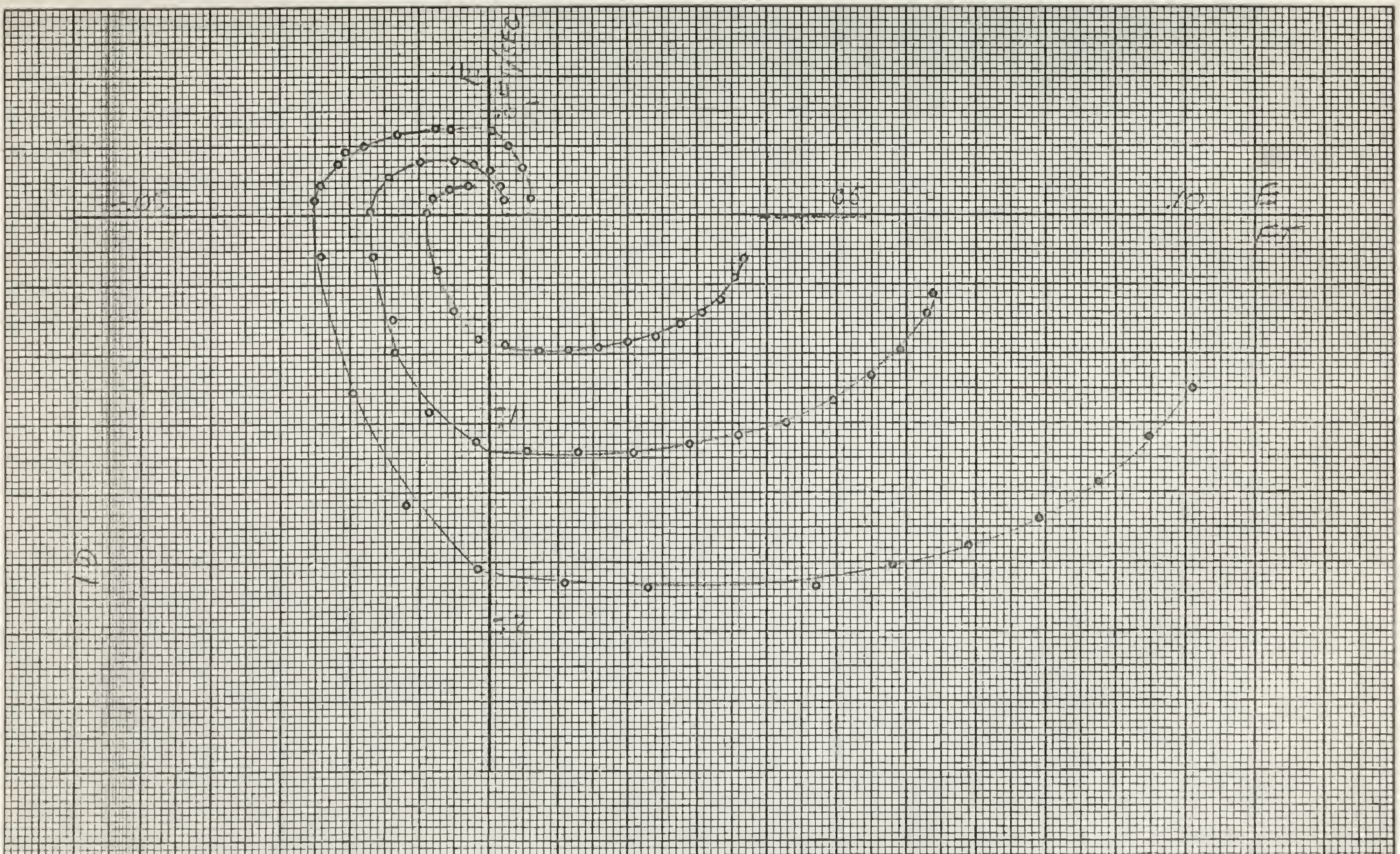


Fig. 7-10 Phase Portrait – Mod. Switching 90% RETURN 0.05 Damping Ratio

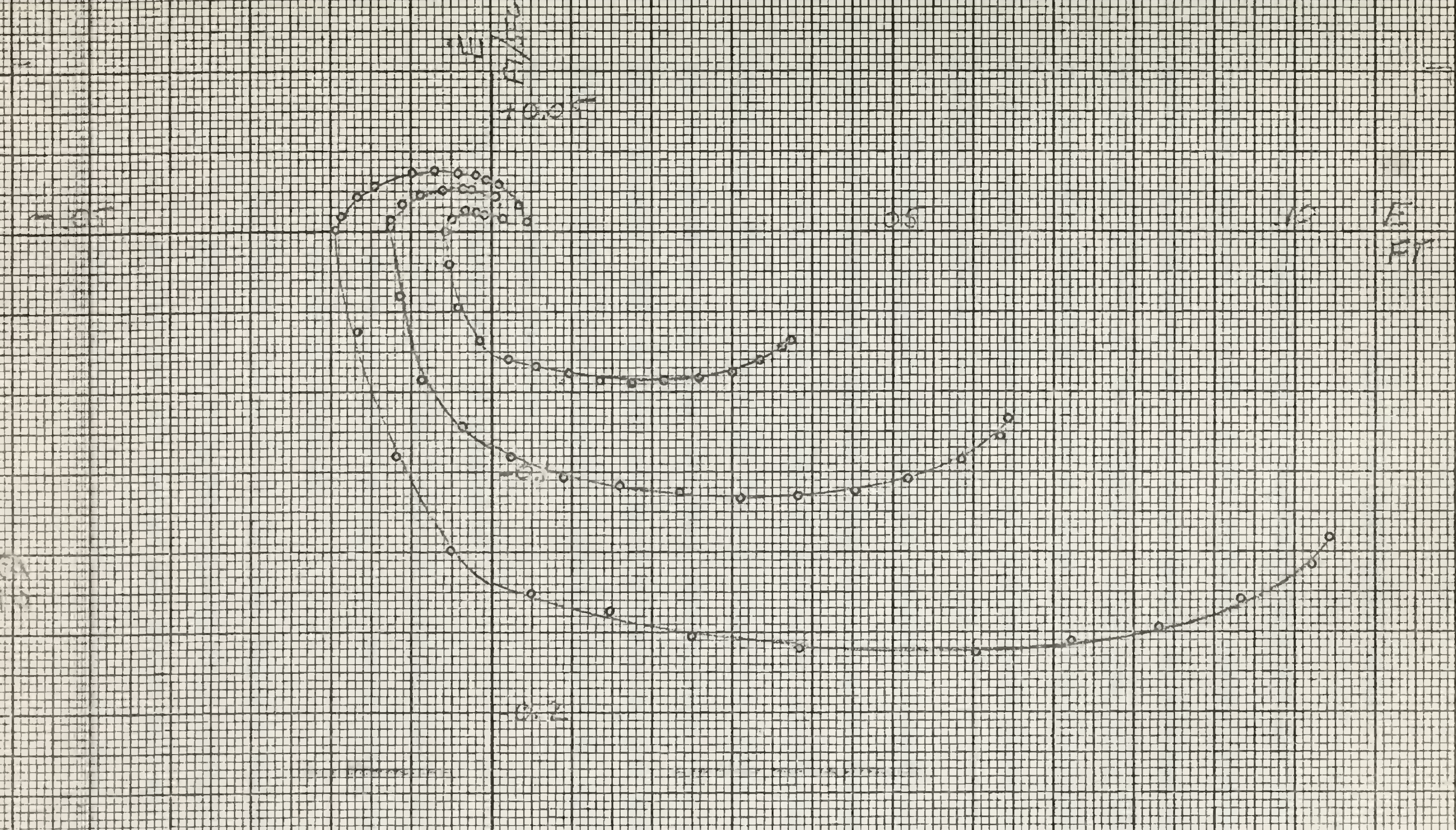


Fig 7-11 Phase Portrait - Modified Switching - 90% Return - 0.1 Damping Ratio

CHAPTER VIII

SUMMARY OF FEASIBILITY STUDY

The experimental work accomplished thus far does not constitute a complete feasibility study of the two-rate spring. Further frequency response tests and step response tests should be made and this work should be extended to include the effects of other components that contribute to "passenger suspension" such as tires, and seat cushions. The results of the frequency response tests and step response tests reported in the previous chapters are encouraging.

Frequency response and transient response characteristics both demonstrate the significant damping action introduced by spring rate switching. Although spring-rate return appears to provide insufficient damping to permit the complete removal of viscous dampers at least viscous damping can be reduced to make step like road imperfections less perceptible to the passenger. The two rate spring also improves ride quality by shifting the automobile body resonant frequency toward the less perceptible end of the comfort spectrum.

The following work is suggested:

1. Conduct further frequency response tests for small damping ratios and large spring rate return using the modified switching criterion.

2. Extend this work to the more complete and more complex system including the unsprung wheel mass and spring-rate of the tire and perhaps the spring rate and damping of the seat cushion. This would involve at least two degrees of

freedom and would permit investigation of wheel resonance (See Figure 3-1).

3. Determine the response of the two-rate spring suspension to ramp inputs and combined inputs such as a positive step followed by a negative step, a negative ramp followed by a positive ramp, etc. Statistical inputs could also be considered.

4. Include a non-linear damper with the two-rate spring such as proposed by Martin and Jeska.[1] The attempt here would be to completely eliminate the initial value velocity associated with a step input. (See Equation 7-2).

5. Devise a mechanical method of accomplishing spring-rate switching on a real spring.

---

[1] Ibid.

BIBLIOGRAPHY

1. G. J. Martin, and R. D. Jeska, IRE Convention Record, PT I, pp 89-98, 1953.

2. H. L. Cox, The Riding Qualities of Wheeled Vehicles, Proceedings of Auto Division of Inst. of Mech.Engineers, 1955-56.

3. David Hodkin, Progress in Comfort, Part One, The Autocar, pp 747-750, May 12, 1961.

4. R. Schilling and H. O. Fuchs, Modern Passenger-Car Ride Characteristics, J. Appl. Mech., 8, pp A-59-65, June, 1941.

5. Lippert, Human Response to Vertical Vibration, SAE Journal, 55, pp 32-34, May 1947.

6. R. N. Janeway, Passenger Vibration Limits, SAE Journal, 56, pp 48-49, August 1948.

7. G. J. Thaler and M. P. Pastel, Analysis and Design of Non-linear Control Systems. McGraw-Hill Book Company, Inc., New York, 1962.

APPENDIX I

Derivation of the Describing Function for the Two-rate Spring

The general Fourier expression is:

$$F(\omega t) = A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots\cdots$$
$$+ B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots\cdots$$

Where $A_1 = 1/\pi \int_0^{2\pi} F(\omega t) \sin \omega t \, d\omega t$

$$B_1 = 1/\pi \int_0^{2\pi} F(\omega t) \cos \omega t \, d\omega t$$

From basic definition of the describing function we consider only the fundamental frequency terms.

$$\therefore \quad F(\omega t) = KE \sin \omega t \qquad 0 < \omega t < \pi/2$$

$$= (K-B) E \sin \omega t \qquad \pi/2 < \omega t < \pi$$

$$A_1 = 1/\pi \left\{ KE \left[\int_0^{\pi/2} \sin^2 \omega t \, d\omega t + \int_{\pi}^{3\pi/2} \sin^2 \omega t \, d\omega t \right] + (K-B)E \left[\int_{\pi/2}^{\pi} \sin^2 \omega t \, d\omega t + \int_{3\pi/2}^{2\pi} \sin^2 \omega t \, d\omega t \right] \right\}$$

$$A_1 = E(K - B/2)$$

$$B_1 = 1/\pi \left\{ KE \left[\int_0^{\pi/2} \sin \omega t \cos \omega t \, d\omega t + \int_{\pi}^{3\pi/2} \sin \omega t \cos \omega t \, d\omega t \right] + (K-B) E \left[\int_{\pi/2}^{\pi} \sin \omega t \cos \omega t \, d\omega t + \int_{\pi}^{3\pi/2} \sin \omega t \cos \omega t \, d\omega t \right] \right\}$$

$$B_1 = EB/\pi$$

$$G_D = \frac{\text{SPRING OUTPUT SIGNAL}}{\text{SPRING INPUT SIGNAL}} = \frac{A_1 \sin \omega t + B_1 \cos \omega t}{E \sin \omega t}$$

$$= \frac{E\left\{(k - B/2)\sin \omega t + B/\pi \cos \omega t\right\}}{E \sin \omega t}$$

$$= \frac{\sqrt{(B/\pi)^2 + (k - B/2)^2}\ \sin\left(\omega t + \tan^{-1} 2B/\pi(2k-B)\right)}{\sin \omega t}$$

$$= \sqrt{(B/\pi)^2 + (k - B/2)^2}\ \epsilon^{j\psi}$$

$$\psi = \tan^{-1} \frac{2B}{\pi(2k-B)}$$

APPENDIX II

Calculation of $E(t)$ and $\dot{E}(t)$ at $t=0$ by application of the Initial Value Theorem

$$\frac{X_2}{X_1} = \left[\frac{S + K/C}{S^2 + (C/M)S + K/M}\right] C/M$$

For a step input $X_1 = A/S$

$$X_2 = \frac{(S + K/C)\,C/M}{S^2 + (C/M)S + K/M}\left(A/S\right)$$

$$X_2(t) \text{ at } t=0 = \lim_{S\to\infty} X_2(S)\,S$$

$$= \lim_{S\to\infty} \frac{(S + K/C)(C/M)\,A}{S^2 + (C/M)S + K/M}$$

$$= \lim_{S\to\infty} \frac{(1/S + K/CS^2)\,C/M\,A}{1 + C/MS + K/MS^2} = 0$$

$$\therefore E(t) \text{ at } t=0 = X_1(t) - X_2(t) = A$$

$$\dot{X}_2(t) = \lim_{S\to\infty} S^2 X_2(S) = \lim_{S\to\infty} \frac{(1 + K/CS)(C/M)A}{1 + C/MS + K/MS^2}$$

$$= (C/M)\,A$$

$$\therefore \dot{E}(t) \text{ at } t=0 = \dot{X}_2(t) = (C/M)\,A$$